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Suggested Solution for Project 1

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1. Let $I = \{1, 2, ..., m\}$ and $J = \{1, 2, ..., n\}$ be the index sets of stores and locations and

$$x_j = \begin{cases} 1 & \text{if a facility is built at location } j \\ 0 & \text{otherwise} \end{cases}, j \in J,$$

$$y_{ij} = \begin{cases} 1 & \text{if store } i \text{ is served by a facility built at location } j \\ 0 & \text{otherwise} \end{cases}, i \in I, j \in J$$

be the decision variables, the problem can be formulated as

$$\begin{aligned} \min \quad & \sum_{i \in I} \sum_{j \in J} h_i d_{ij} y_{ij} + \sum_{j \in J} f_j x_j \\ \text{s.t.} \quad & \sum_{j \in J} y_{ij} = 1 \quad \forall\, i \in I \\ & y_{ij} \leq x_j \quad \forall\, i \in I, j \in J \\ & x_j, y_{ij} \in \{0, 1\} \quad \forall\, i \in I, j \in J \end{aligned}$$

The first constraint requires each customer to be assigned to at least one facility. The second constraint allows y_{ij} to be 1, which means facility j serves store i, only if there is a facility built at location j. With the minimization objective function, the y_{ij} with the minimum d_{ij} among all available facilities will be set to 1. The objective function minimizes the sum of shipping cost (the first part) and the construction cost (the second part).

2. First we show that adopting linear relaxation of y_{ij} still gives us an optimal integer solution (i.e., even if you replace $y_{ij} \in \{0, 1\}$ by $y_{ij} \in [0, 1]$, any optimal solution you get will still satisfy $y_{ij} \in \{0, 1\}$. Assume all d_{ij}'s are different for a while. Before we relax this constraint, the minimization objective has already found a location j^*, among all open facilities, with the minimum $h_i d_{ij}$ and set $y_{i,j^*} = 1$. Suppose we replace $y_{ij} \in \{0, 1\}$ by $y_{ij} \in [0, 1]$. If the solution changes from $y_{i,j^*} = 1$ to $y_{i,j^*} = r < 1$, the objective value must increase since $h_i d_{i,j^*} < h_i d_{ij}$ for all $j \neq j^*$. If some d_{ij}s are identical, there may be optimal solutions with fractional $Yyij$s. Nevertheless, there must still be an optimal solution with binary y_{ij}s. Such an optimal solution with binary y_{ij}s will be found because for any linear (integer) program, only extreme point solutions will be reported as optimal solutions.

   If y_{ij}s are fixed to be binary, we may also relax the integer constraint of x_js because if x_j can be 0, setting it to be any fractional number larger than 0 is suboptimal; if x_j must be 1, setting it to be any fractional number less than 1 is infeasible.

   However, we cannot relax the integer constraint for both x_js and y_{ij}s. One may easily generate examples that the only optimal solution requires some variables to be fractional.

3. As it is believed that a polynomial-time exact algorithm (which guarantees to find an optimal solution) does not exist, there is no way to claim that your algorithm is "wrong" as long as that makes some sense. All you need to do is to intuitively explain the rationales behind your algorithm and why intuitive it finds good solutions. If you provide any analysis or conjecture about the performance of your algorithm for different kinds of problems, you will get higher grades.