IM2010: Operations Research
Game Theory: Dynamic Games

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Road map

- **Dynamic games**.
- Pricing in a supply chain.

Dynamic BoS

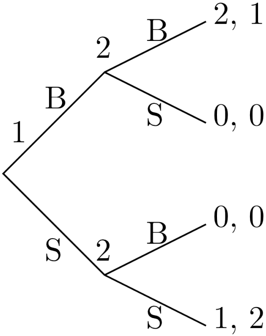
- Recall the game "Bach or Stravinsky":

	Bach	Stravinsky
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

- What if the two players make decisions **sequentially** rather than simultaneously?
  - What will they do in equilibrium?
  - How do their payoffs change?
  - Is it better to be the **leader** or the **follower**?

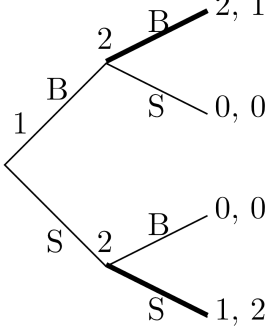
Dynamic BoS

- Suppose player 1 **moves** first.
- Instead of a game matrix, the game can now be described by a **game tree**.
  - At each internal node, the label shows who is moving.
  - At each link, the label shows an action.
  - At each leaf, the numbers show the payoffs.
- The games is played from the root to leaves.



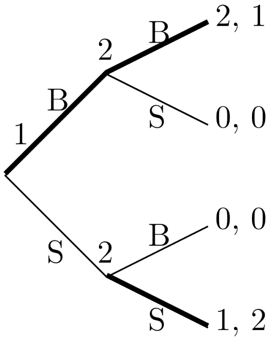
Dynamic BoS: Player 2's strategy

- How should player 1 move?
  - She needs to first **predict** how player 2 will response.
- She first treats herself as player 2:
  - If B has been chosen, choose B.
  - If S has been chosen, choose S.
- This is exactly player 2's **best response** to player 1's action.
  - It is also player 2's optimal strategy.
- We use thick lines to mark player 2's optimal strategy.



Dynamic BoS: Player 1's strategy

- How should player 1 move?
  - She knows how player 2 reacts.
  - Based on that, she chooses her action.
- Player 1 thinks:
  - If I choose B, I will end up with 2.
  - If I choose S, I will end up with 1.
- So player 1 will choose B.
- We also use a thick line to mark player 1's optimal strategy.
- A thick line that connects the root and a leave is an **equilibrium outcome**.
  - In equilibrium, they play (B, B).



Dynamic BoS vs. static BoS

- In static BoS, there are three (mixed-strategy) Nash equilibria.
  - Two of them are pure-strategy: (B, B) and (S, S).
- Regarding predicting their behaviors:
  - In the static case, we cannot perfectly predict what they will do.
  - But in the dynamic case, we can!
  - Their **equilibrium behaviors** change. Is it always the case?
- What if player 2 is the leader and player 1 is the follower?

Dynamic prisoners' dilemma

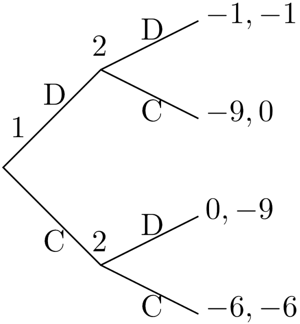
- Recall the game "prisoners' dilemma":

	Denial	Confession
Denial	$-1, -1$	$-9, 0$
Confession	$0, -9$	$-6, -6$

- The equilibrium outcome is (Denial, Denial).
  - This is due to the lack of coordination.
  - In particular, they cannot communicate and cannot observe what the other player chooses.
- Will the outcome change when they move sequentially?

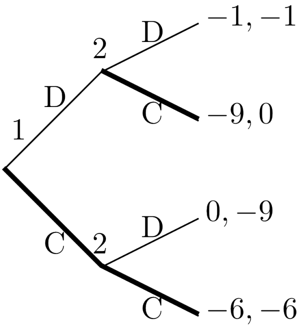
Dynamic prisoners' dilemma

- Suppose player 1 moves first.
- The game tree is depicted here.
- Again, before player 1 makes her decision, she must predict what player 2 will do.
- What will they do in equilibrium?



Dynamic prisoners' dilemma

- Player 2's optimal strategy:
  - If she denies, I should confess.
  - If she confesses, I should confess.
- Player 1's optimal strategy:
  - If I denies, I will end up with -9.
  - If I confess, I will end up with -6.
- In equilibrium, they will both confess.
  - The outcome does not change!
  - Even if they have agreed to both deny once they are caught, even if player 1 has denied and player 2 has observed it, player 2 will still confess.

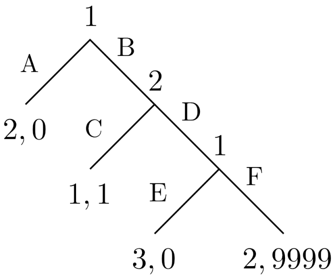


Backward induction

- In the previous two examples, there are a leader and a follower.
- Before the leader can make her decision, she must anticipate what the follower will do.
- In general, when there are multiple **stages** in a **dynamic game**, we analyze those decision problems **from the last stage**.
  - Then the second last stage problem can be solved by having the last stage behavior in mind.
  - The the third last stage problem can be solved.
  - We move **backwards** until the first stage problem is solved.
- This solution concept is called **backward induction**.

A three-stage dynamic game

- Consider the three-stage game depicted below:



  - In this game, player 1 has two moves: at stage 1 and at stage 3.
  - Player 2 has only one move: at stage 2.
- What will be the equilibrium outcome?

A discussion on rationality

- Is it really the case that "when player 2 has the chance to act, she will always choose C"?
  - If player 1 is **rational**, player 2 should never get a chance to act.
  - If player 2 gets a chance to act, that somehow means player 1 is not completely rational.
  - Therefore, if player 2 chooses D, it is **possible** for player 1 to choose F.
  - So player 2 should not completely abandon D.
- **Bounded rationality** has been studied in various subjects.
  - We will not touch it in this course.

Leader's advantage

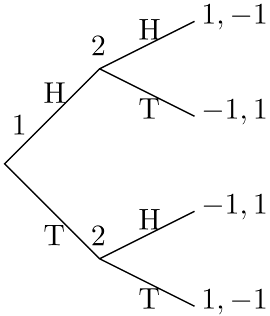
- In BoS, being the leader (who acts first) is beneficial.
- In prisoners' dilemma, being the leader or not does not matter.
- In most chess games, being the leader is advantageous.
- Is it always a good idea to be the leader?

Dynamic matching pennies

- Recall the game "matching pennies":

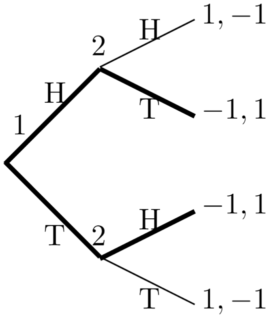
	Head	Tail
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

- What is the equilibrium outcome?



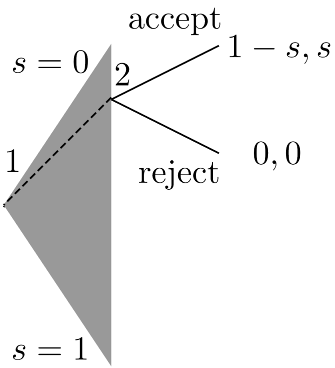
Dynamic matching pennies

- In equilibrium, player 1 is always dominated by player 2:
  - Player 2 will choose whatever player 1 does not choose.
  - It does not matter how player 1 acts
- There are multiple possible outcomes.
- Being the leader **hurts** player 1.



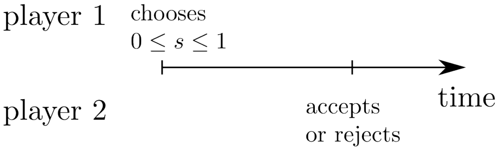
The ultimatum game

- We conclude this section with the classical ultimatum game.
  - This is an example with an infinite action space.
- In an ultimatum game:
  - Player 1 decides how to share \$1 with player 2 by offering him \$$s$.
  - Player 2 may accept or reject the offer.
  - If he accepts, he earns \$$s$ and player 1 earns \$$(1 - s)$.
  - If he rejects, both of them earns \$0.
- Suppose both of them are completely rational and want to maximize their payoffs. What will they do?



The time line representation

- In many cases (e.g., when a player has an infinite action space), it is a good idea to use a time line to illustrate the timing of a dynamic game.



The ultimatum game

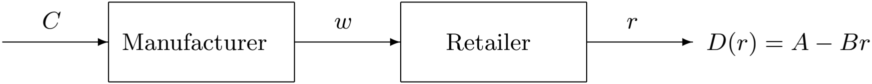
- In equilibrium, player 1 earns \$1 and player 2 earns \$0!
  - In practice, it may be player 1 earning $\$(1-\epsilon)$ and player 2 earning $\$\epsilon$ for some $\epsilon > 0$.
  - Theoretically, however, only (0, accept) and (0, reject) may be equilibrium outcomes.
- This applies to many real-world cases:
  - E.g., wage negotiation between an employer and a employee.
- How may we modify this game to achieve a fair allocation (to make both players earn \$0.5)?

Road map

- Dynamic games.
- **Pricing in a supply chain**.

Pricing in a supply chain

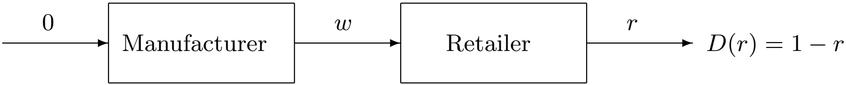
- There is a manufacturer and a retailer in a supply chain.



  - The manufacturer produces and supplies to the retailer. The retailer sells to end consumers.
  - The manufacturer sets the **wholesale price** w and then the retailer sets the **retail price** r.
- The demand is $D(r) = A - Br$, where A and B are known constants.
- The unit production cost is C, a known constant.

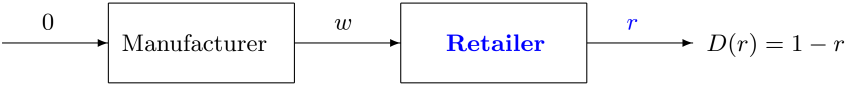
Pricing in a supply chain

- What is the equilibrium (i.e., what will the two players do)?
- We call an equilibrium as a **solution** of a game.
- To make our lives easier, let's assume $A = B = 1$ and $C = 0$.



- Let's apply backward induction to **solve** this game.

Pricing in a supply chain (illustrative)

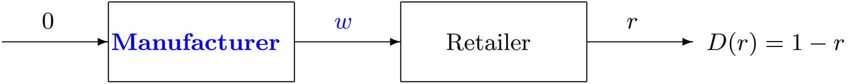


- For the retailer, the wholesale price is **given**. His trade off:
  - Making price lower decreases the profit margin $r - w$.
  - Making price higher decreases the sales volume $1 - r$.
- The retailer's problem:

$$\begin{aligned} & \max\ (r-w)(1-r) \\ = & \max\ -r^2 + (w+1)r - w \end{aligned}$$

- The optimal solution (best response) is $r^*(w) = \frac{w+1}{2}$.

Pricing in a supply chain (illustrative)



- The manufacturer **predicts** the retailer's decision:
  - Given her offer w, the retail price will be $r^*(w) = \frac{w+1}{2}$.
  - More importantly, the **order quantity** will be

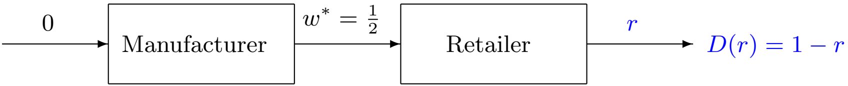
$$1 - r^*(w) = 1 - \frac{w+1}{2} = \frac{1-w}{2}.$$

  - The manufacturer's problem:

$$\max \; w\left(\frac{1-w}{2}\right) = \max \; \frac{-w^2+w}{2}.$$

  - The optimal solution is $w^* = \frac{1}{2}$.

Pricing in a supply chain (illustrative)

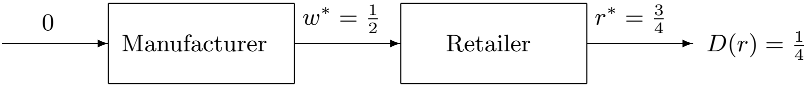


- Given that the manufacturer will offer the wholesale price $w^* = \frac{1}{2}$, the resulting retail price will be

$$r^* \equiv r^*(w^*) = \frac{w^* + 1}{2} = \frac{\frac{1}{2} + 1}{2} = \frac{3}{4} > \frac{1}{2} = w^*.$$

  - A common practice called **markup**.
- The **sales volume** is $D(r^*) = 1 - r^* = \frac{1}{4}$.

Pricing in a supply chain (illustrative)



- The retailer earns
$$(r^* - w^*)D(r^*) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16}.$$
- The manufacturer earns
$$(w^* - C)D(r^*) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{8}.$$
- In total, they earn
$$\frac{1}{16} + \frac{1}{8} = \frac{3}{16}.$$

Pricing in a supply chain (general)

- For the retailer, the wholesale price is given and fixed.
- His trade off:
  - Making price lower decreases the profit margin $w - r$.
  - Making price higher decreases the sales volume $A - Br$.
- The retailer's problem:

$$\begin{aligned} & \max\ (r-w)(A-Br) \\ = & \max\ -Br^2 + (Bw + A)r - Aw \end{aligned}$$

- The optimal solution is $r^*(w) = \dfrac{Bw + A}{2B}$.

Pricing in a supply chain (general)

- The manufacturer predicts the retailer's decision:
  - Given her offer w, the retail price will be $r^*(w) = \frac{Bw+A}{2B}$.
  - More importantly, the order quantity will be

$$A - Br^*(w) = A - \frac{Bw + A}{2} = \frac{A - Bw}{2}.$$

- The manufacturer's problem:

$$\begin{aligned} & \max\ (w - C)\left(\frac{A - Bw}{2}\right) \\ = & \max\ \frac{-Bw^2 + (BC + A)w - AC}{2} \end{aligned}$$

- The optimal solution is $w^* = \frac{BC + A}{2B}$.

Pricing in a supply chain (general)

- Given that the manufacturer will offer the wholesale price $w^* = \frac{BC+A}{2B}$, the resulting retail price will be

$$r^* \equiv r^*(w^*) = \frac{Bw^* + A}{2B} = \frac{\frac{BC+A}{2} + A}{2B} = \frac{BC + 3A}{4B}.$$

- The sales volume is $D(r^*) = A - Br^* = \frac{A-BC}{4}$.

Pricing in a supply chain (general)

- The retailer earns

$$(r^* - w^*)D(r^*) = \left(\frac{A - BC}{4B}\right)\left(\frac{A - BC}{4}\right) = \frac{(A - BC)^2}{16B}.$$

- The manufacturer earns

$$(w^* - C)D(r^*) = \left(\frac{A - BC}{2B}\right)\left(\frac{A - BC}{4}\right) = \frac{(A - BC)^2}{8B}.$$

- In total, they earn

$$\frac{(A - BC)^2}{16B} + \frac{(A - BC)^2}{8B} = \frac{3(A - BC)^2}{16B}.$$

Pricing in a cooperative supply chain

(Figure source: http://www.property.al/2009/03/the-property-purchase-process-in-albania/)

- Suppose the two firms are **cooperative**, i.e., they sit down and discuss what to do together.
- They can decide the wholesale and retail prices together.
- However, they must make sure that both players **do better** than when the supply chain is decentralized.
- Any idea?

Pricing in a cooperative supply chain

- Consider the following proposal:
  - Let's set $w^{FB} = C = 0$ and $r^{FB} = \frac{1}{2}$.
  - The sales volume is

$$D(r^{FB}) = 1 - \frac{1}{2} = \frac{1}{2}.$$

  - The total profit is

$$r^{FB} D(r^{FB}) = \frac{1}{4}.$$

  - This is **larger** than $\frac{3}{16}$, the total profit generated under decentralization.
  - We then **split this pie**!

Pricing in a cooperative supply chain

- How to split the pie?
- Recall that the manufacturer earns $\frac{1}{8}$ and the retailer earns $\frac{1}{16}$ under decentralization.
- So how about this:
  - First the manufacturer gets $\frac{1}{8}$.
  - Then the retailer gets $\frac{1}{16}$.
  - Then each of us gets the remaining $\frac{1}{16}$.
- **Win-win**!

Efficiency v.s. Inefficiency

- When the supply chain is not cooperative, it is operated under **decentralization**.
- When the supply chain is cooperative or controlled by a single central planner, it is under **centralization**.
- Centralization always results in a socially optimal solution.
  - A socially optimal solution is called the "**first best**" solution.
  - Only if the planner is smart ...
  - And the distribution of wealth can be a problem.
  - But anyway, cooperation is generally good.
- Decentralization often results in **efficiency loss**.
  - The efficiency loss in this example is $\frac{1}{4} - \frac{3}{16} = \frac{1}{16}$.