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Suggested Solution for Homework 2

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1. (a) Since $(0, ..., 0, 1)$ can be obtained by a combination of $(0, ..., 0, 0)$ and $(0, ..., 0, 2)$ as $(0, ..., 0, 1) = \frac{1}{2}(0, ..., 0, 0) + \frac{1}{2}(0, ..., 0, 2)$, it is not an extreme point.

   (b) For point 0, if there exists two points x and y such that $\lambda x + (1 - \lambda)y = 0$, either x or y must be negative. As all the feasible points are nonnegative, point 0 is an extreme point. For any other feasible point n (which is a positive integer), it can be obtained by a combination of $n-1$ and $n+1$ as $\frac{1}{2}(n-1) + \frac{1}{2}(n+1) = i$. Therefore, no positive integer is an extreme point.

2. The feasible region and isoquant line are illustrated in Figure 1. It is clear that we should push the isoquant line until we stop at the extreme point $(2, 2)$, which is an optimal solution.

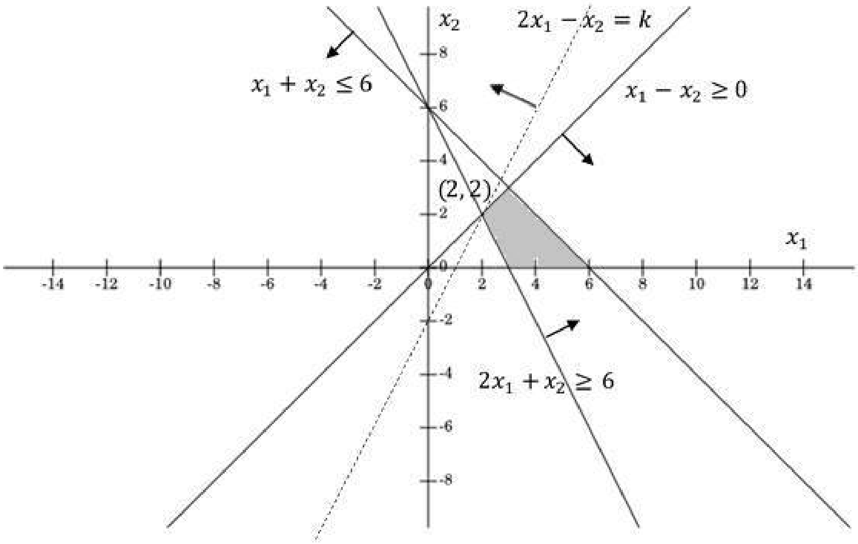

Figure 1: Graphical solution for Problem 2

3. Omitted.

4. To make our notation concise, we label Monday as day 1, Tuesday as day 2, ..., and Sunday as day 7. With the labeling, let the decision variables be

$$x_{ij} = \text{number of officers whose days off are on days } i \text{ and } j,\ i = 1, ..., 6, j = i + 1, ..., 7.$$

The objective is to minimize the number of officers whose days off are not consecutive, or equivalently, maximizing the number of officers whose days off are consecutive. Therefore, we maximize

$$x_{12} + x_{23} + \cdots + x_{67} + x_{17}.$$

For Monday, we need at least 12 officers, which means we may have at most 18 officers off on Monday. This is achieved by having

$$x_{12} + x_{13} + \cdots + x_{17} \leq 18.$$

Similar arguments give us the constraints for the other six days. Finally, the total number of officers is 30, so we should have

$$x_{12} + x_{13} + \cdots + x_{67} = 30.$$

The complete formulation is

$$
\begin{aligned}
z^* = \max\quad & x_{12} + x_{23} + x_{34} + x_{45} + x_{56} + x_{67} + x_{17} \\
\text{s.t.}\quad & x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} \le 18 && \text{(Number of officers off on Monday)} \\
& x_{12} + x_{23} + x_{24} + x_{25} + x_{26} + x_{27} \le 10 && \text{(Number of officers off on Tuesday)} \\
& x_{13} + x_{23} + x_{34} + x_{35} + x_{36} + x_{37} \le 12 && \text{(Number of officers off on Wednesday)} \\
& x_{14} + x_{24} + x_{34} + x_{45} + x_{46} + x_{47} \le 8 && \text{(Number of officers off on Thursday)} \\
& x_{15} + x_{25} + x_{35} + x_{45} + x_{56} + x_{57} \le 5 && \text{(Number of officers off on Friday)} \\
& x_{16} + x_{26} + x_{36} + x_{46} + x_{56} + x_{67} \le 5 && \text{(Number of officers off on Saturday)} \\
& x_{17} + x_{27} + x_{37} + x_{47} + x_{57} + x_{67} \le 14 && \text{(Number of officers off on Sunday)} \\
& \textstyle\sum_{i=1}^{6} \sum_{j=i+1}^{7} x_{ij} = 30 && \text{(Total number of officers)} \\
& x_{ij} \ge 0 \quad \forall\, i = 1, ..., 6, j = i+1, ..., 7.
\end{aligned}
$$

5. Let the decision variables be

$$x_i = \text{pounds of chemical } i \text{ used, } i = 1, ..., 4.$$

The complete formulation of this problem is

$$
\begin{array}{lrcrcrcrcll}
\min & 8x_1 & + & 12x_2 & + & 13x_3 & + & 15x_4 & & & \\
\text{s.t.} & x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1000 & \text{(Total amount produced)} \\
& 0.04x_1 & + & 0.06x_2 & + & 0.1x_3 & + & 0.11x_4 & \ge & 50 & \text{(Quality: ingredient A)} \\
& 0.02x_1 & + & 0.05x_2 & + & 0.03x_3 & + & 0.09x_4 & \ge & 40 & \text{(Quality: ingredient B)} \\
& 0.01x_1 & + & 0.01x_2 & + & 0.03x_3 & + & 0.04x_4 & \ge & 20 & \text{(Quality: ingredient C)} \\
& & & x_2 & & & & & \ge & 100 & \text{(Least amount of chemical 2)} \\
& & & & & & & x_i & \ge & 0 & \forall\, i = 1, ..., 4.
\end{array}
$$

The three quality constraints on A, B, and C are formulated by making some percentages large enough. For example, for A we must have

$$\frac{0.04x_1 + 0.06x_2 + 0.1x_3 + 0.11x_4}{x_1 + x_2 + x_3 + x_4} \ge 0.05. \tag{1}$$

As we emphasized in a lecture, we must remove a nonlinear formulation like this by moving the denominator to the RHS. Because the total amount produced must be exactly 1000 lb, i.e., $x_1 + x_2 + x_3 + x_4 = 1000$, we convert the nonlinear formulation into a linear one: $0.04x_1 + 0.06x_2 + 0.1x_3 + 0.11x_4 \ge 0.05 \times 1000 = 50$. The quality constraints on B and C are also obtained in the same way.

6. Let the decision variables be

$$x_{ij} = \text{ounces of chemical } j \text{ used to produce drug } i,\ i = 1, ..., m,\ j = 1, ..., n.$$

The complete formulation of this problem is

$$
\begin{aligned}
\max\quad & \sum_{i=1}^{m} \sum_{j=1}^{n} P_i x_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{n} C_j x_{ij} \\
\text{s.t.}\quad & x_{ij} \ge M_{ij} \sum_{k=1}^{n} x_{ik} \quad \forall\, i = 1, ..., m, j = 1, ..., n \\
& \sum_{j=1}^{n} x_{ij} \le D_i \quad \forall\, i = 1, ..., m \\
& \sum_{i=1}^{m} x_{ij} \le S_j \quad \forall\, j = 1, ..., n \\
& x_{ij} \ge 0 \quad \forall\, i = 1, ..., m, j = 1, ..., n.
\end{aligned}
$$

7. To make our notation concise, we label the two refineries at Kaohsiung and Taipei as refinery 1 and 2 and the two distribution points at Hsinchu and Taichung as distribution points 1 and 2. Then for $i = 1, 2$ and $j = 1, 2$, we define the decision variables as

$$\begin{aligned} w_{ij} &= \text{million barrels of "original" capacity shipped from refinery } i \text{ to distribution point } j, \text{ and} \\ z_{ij} &= \text{million barrels of "additional" capacity shipped from refinery } i \text{ to distribution point } j. \end{aligned}$$

For parameters, we denote P_{ij} as the profit (in thousand dollars) per million barrels of oil shipped from refinery i to distribution point j, C_i as the unit cost (in thousands) of additional capacity for one million barrel in refinery i, K_i as the current capacity (in million barrel) in refinery i, and D_j as the demand size (in million barrels) at distribution point j for all $i = 1, 2$ and $j = 1, 2$. With the definitions of variables and parameters, we formulate the problem as

$$\begin{aligned} \max \quad & 10\sum_{i=1}^{2}\sum_{j=1}^{2} P_{ij}(w_{ij} + z_{ij}) - \sum_{i=1}^{2}\sum_{j=1}^{2} C_i z_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{2}(w_{ij} + z_{ij}) \leq D_j \quad \forall\, j = 1, 2 \\ & \sum_{j=1}^{2} w_{ij} \leq K_i \quad \forall\, i = 1, 2 \\ & w_{ij}, z_{ij} \geq 0 \quad \forall\, i = 1, 2, j = 1, 2. \end{aligned}$$

The objective function consists of two parts, the 10-year total profit and the one-time expansion cost. The first constraint ensures that the total sales at each distribution point is at most the demand size. The second constraint ensures that the total production quantity at each refinery does not excess the post-expansion capacity. The last constraint is the nonnegativity constraint.

8. Omitted.