Operations Research

Lab Session

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Outline

1. Homework 3 illustration
2. Simplex method (two phase)
3. LP model for max function and absolute value

Problem 1

$$\begin{aligned} \min \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 \geq 3 \\ & x_1 + x_2 \geq -4 \\ & 2x_1 - x_2 = 3 \\ & x_1 \leq 0, x_2 \text{ urs} \end{aligned}$$

Requirement 1: Nonnegative RHS

$$\begin{aligned} \min \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 \geq 3 \\ & x_1 + x_2 \geq \boxed{-4} \\ & 2x_1 - x_2 = 3 \\ & x_1 \leq 0, x_2 \text{ urs} \end{aligned}$$

=>

$$\begin{aligned} \min \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 \geq 3 \\ & -x_1 - x_2 \leq 4 \\ & 2x_1 - x_2 = 3 \\ & x_1 \leq 0, x_2 \text{ urs} \end{aligned}$$

Problem 1

Requirement 2: Nonnegative variables

$$\begin{aligned} \min\quad & 3x_1 + x_2 \\ \text{s.t.}\quad & x_1 \geq 3 \\ & -x_1 - x_2 \leq 4 \\ & 2x_1 - x_2 = 3 \\ & x_1 \leq 0, x_2 \text{ urs} \end{aligned} \quad => \quad \begin{aligned} \min\quad & -3x_1 + x_2 \\ \text{s.t.}\quad & -x_1 \geq 3 \\ & x_1 - x_2 \leq 4 \\ & -2x_1 - x_2 = 3 \\ & x_1 \geq 0, x_2 \text{ urs} \end{aligned}$$

$$\begin{aligned} \min\quad & -3x_1 + x_2 \\ \text{s.t.}\quad & -x_1 \geq 3 \\ & x_1 - x_2 \leq 4 \\ & -2x_1 - x_2 = 3 \\ & x_1 \geq 0, x_2 \text{ urs} \end{aligned} \quad => \quad \begin{aligned} \min\quad & -3x_1 + x_2' - x_2'' \\ \text{s.t.}\quad & -x_1 \geq 3 \\ & x_1 - x_2' + x_2'' \leq 4 \\ & -2x_1 - x_2' + x_2'' = 3 \\ & x_1, x_2', x_2'' \geq 0 \end{aligned}$$

Problem 1

Requirement 3: Equality constraints

$$\begin{aligned} \min \quad & -3x_1 + x_2' - x_2'' \\ \text{s.t.} \quad & -x_1 \geq 3 \\ & x_1 - x_2' + x_2'' \leq 4 \\ & -2x_1 - x_2' + x_2'' = 3 \\ & x_1, x_2', x_2'' \geq 0 \end{aligned}$$

=>

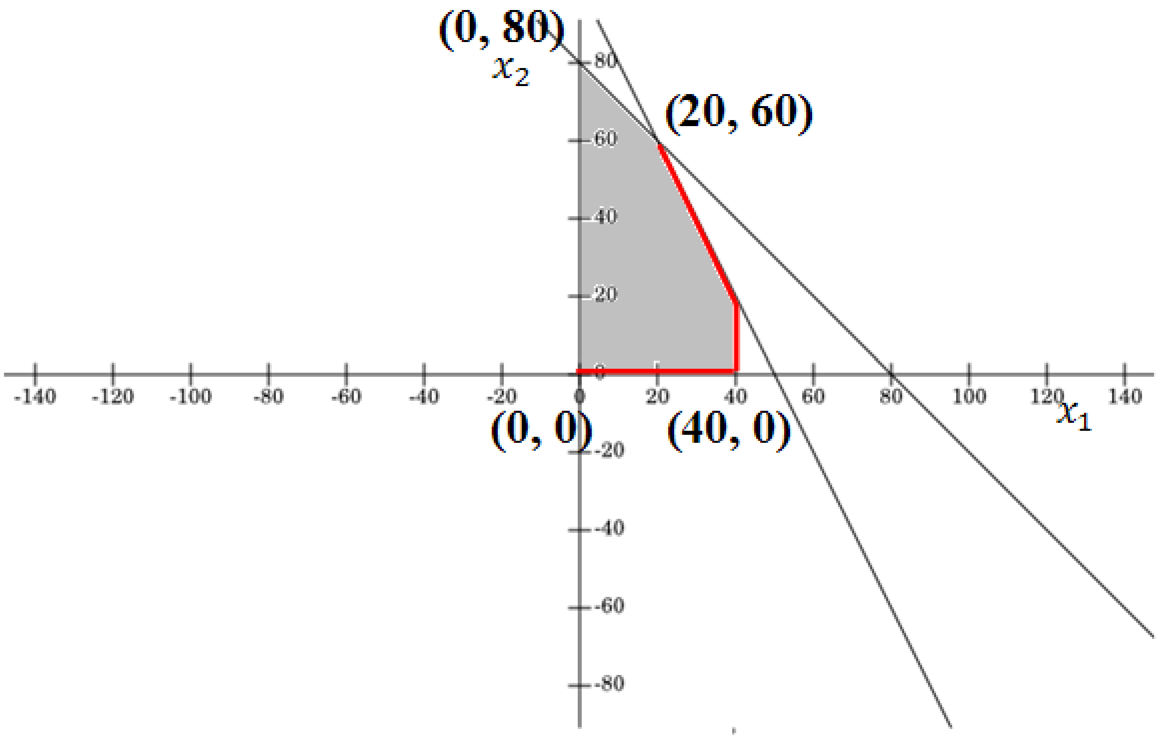
$$\begin{aligned} \min \quad & -3x_1 + x_2' - x_2'' \\ \text{s.t.} \quad & -x_1 - x_3 = 3 \\ & x_1 - x_2' + x_2'' \leq 4 \\ & -2x_1 - x_2' + x_2'' = 3 \\ & x_1, x_2', x_2'', x_3 \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & -3x_1 + x_2' - x_2'' \\ \text{s.t.} \quad & -x_1 - x_3 = 3 \\ & x_1 - x_2' + x_2'' \leq 4 \\ & -2x_1 - x_2' + x_2'' = 3 \\ & x_1, x_2', x_2'', x_3 \geq 0 \end{aligned}$$

=>

$$\begin{aligned} \min \quad & -3x_1 + x_2' - x_2'' \\ \text{s.t.} \quad & -x_1 - x_3 = 3 \\ & x_1 - x_2' + x_2'' + x_4 = 4 \\ & -2x_1 - x_2' + x_2'' = 3 \\ & x_1, x_2', x_2'', x_3, x_4 \geq 0 \end{aligned}$$

Problem 2.4



(0, 80)
x_2
80
60
40
20
0
(20, 60)
-140
-120
-100
-80
-60
-40
-20
0
20
40
60
80
100
120
140
x_1
(0, 0)
(40, 0)
-20
-40
-60
-80

Problem 5

- Variable with the minimum ratio implies become 0 faster and…

1. Since the variable become 0, it can be non-basic variable.
2. It means that will be the first constraint we hit.
3. If we choose bigger ratio, RHS will negative.
4. If we choose bigger ratio, we stay at infeasible region.

Simplex method

Simplex method

- Why we need two phases?
  Phase-I => find a initial point
  Phase-II => find optimal solution

$$
\begin{aligned}
\min \quad & x_1 - 2x_2 \\
\text{s.t.} \quad & x_1 \geq 5 \\
& x_1 + x_2 \leq 10 \\
& x_2 \geq 0
\end{aligned}
$$

It seems we cannot start from (0, 0)!

Simplex method Phase-I

$$\begin{aligned} \min \quad & && && && && +x_5 \quad \text{artificial variable} \\ \text{s.t.} \quad & x_1 && && -x_3 && && +x_5 \geq 5 \\ & x_1 && +2x_2 && && +x_4 && \leq 10 \\ & x_2 \geq 0 \end{aligned}$$

0	0	0	0	-1	0
1	0	-1	0	1	5
1	2	0	1	0	10

Add artificial variables when you see " $\geq$ " and " $=$ ".

Simplex method Phase-I

0	0	0	0	-1	0
1	0	-1	0	1	$5(x_5)$
1	2	0	1	0	$10(x_4)$

1	0	-1	0	0	5
1	0	-1	0	1	$5(x_5)$
1	2	0	1	0	$10(x_4)$

0	0	0	0	-1	0
1	0	-1	0	1	$5(x_1)$
0	2	1	1	-1	$5(x_4)$

Initial point is (5, 0)

Simplex method Phase-II

- Remain the result in phase-I

0	0	0	0	-1	0
1	0	-1	0	1	$5(x_1)$
0	2	1	1	-1	$5(x_4)$

- Remove artificial variable and change objected function

-1	2	0	0	0
1	0	-1	0	$5(x_1)$
0	2	1	1	$5(x_4)$

$$\begin{aligned} \min \quad & x_1 - 2x_2 \\ \text{s.t.} \quad & x_1 \geq 5 \\ & x_1 + x_2 \leq 10 \\ & x_2 \geq 0 \end{aligned}$$

Simplex method Phase-II

-1	2	0	0	0
1	0	-1	0	$5(x_1)$
0	2	1	1	$5(x_4)$

↓

0	2	-1	0	5
1	0	-1	0	$5(x_1)$
0	[2]	1	1	$5(x_4)$

↓

0	0	-2	0	0
1	0	-1	0	$5(x_1)$
0	1	1/2	½	$5/2(x_2)$

$$\begin{aligned} \min \quad & x_1 - 2x_2 \\ \text{s.t.} \quad & x_1 \geq 5 \\ & x_1 + x_2 \leq 10 \\ & x_2 \geq 0 \end{aligned}$$

Practice

$$
\begin{aligned}
\min \quad & 6x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 + x_2 \geq 1 \\
& 2x_1 - x_2 \geq 1 \\
& 3x_2 \leq 2 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Answer

0	0	-4	-1	0	5
0	1	-2/3	1/3	0	1/3(x_2)
1	0	-1/3	-1/3	0	2/3(x_1)
0	0	2	-1	1	1(x_5)

max function and absolute value

Max function

$$\begin{array}{ll} \min & \mathrm{Z} = \max_{i=1,\ldots,m}(c_i^T x) \\ \mathrm{s.t.} & Ax = b \end{array}$$

$$\begin{array}{ll} \min & \mathrm{Z} = \max\,(x_1 + x_2, 2x_1 - x_2) \\ \mathrm{s.t.} & x_1 + 3x_2 \le 5 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{array}$$

$$\begin{array}{ll} \min & Z = y \\ \mathrm{s.t.} & Ax = b \\ & c_i^T x \le y, i = 1, \ldots, m \\ & y \ge 0 \end{array}$$

$$\begin{array}{ll} \min & Z = x_3 \\ \mathrm{s.t.} & x_1 + 3x_2 \le 5 \\ & x_1 \ge 0,\ x_2 \ge 0 \\ & x_1 + x_2 \le x_3 \\ & 2x_1 - x_2 \le x_3 \\ & x_3 \ge 0 \end{array}$$

$x_1 + x_2 \le x_3$ => $x_1 + x_2 - x_3 \le$

Absolute value

$$\begin{aligned} \min \quad & Z = |c^T x| \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

$$\begin{aligned} \min \quad & Z = |2x_1 - x_2| \\ \text{s.t.} \quad & x_1 + 3x_2 \le 7 \\ & x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \min \quad & Z = y \\ \text{s.t.} \quad & Ax = b \\ & c^T x \le y \\ & c^T x \ge -y \\ & y \ge 0 \end{aligned}$$

$$\begin{aligned} \min \quad & Z = y \\ \text{s.t.} \quad & x_1 + 3x_2 \le 7 \\ & x_1 \ge 0,\ x_2 \ge 0 \\ & 2x_1 - x_2 \le y \\ & 2x_1 - x_2 \ge -y \\ & y \ge 0 \end{aligned}$$

Practice

$$\min \quad |2x_1 - x_2|$$

$$\text{s.t.} \quad x_1 + x_2 \geq 3$$

$$x_1 \geq 0,\ x_2 \geq 0$$

Thank you ☺