

Operations Research

Linear Programming Duality

Ling-Chieh Kung

Department of Information Management
National Taiwan University

Introduction

- ▶ For business, we study how to formulate LPs.
- ▶ For engineering, we study how to solve LPs.
- ▶ For science, we study mathematical **properties** of LPs.
 - ▶ We will study **Linear Programming duality**.
 - ▶ It still has important applications.

Road map

- ▶ **Primal-dual pairs.**
- ▶ Duality theorems.
- ▶ Shadow prices.

Upper bounds of a maximization LP

- ▶ Consider the following LP

$$\begin{aligned} z^* = \max \quad & 4x_1 + 5x_2 + 8x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq 6 \\ & 2x_1 + x_2 + 2x_3 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

- ▶ Suppose the LP is very hard to solve.
- ▶ Your friend proposes a solution $\hat{x} = (\frac{1}{2}, 1, 1)$ with $\hat{z} = 15$.
 - ▶ If we know z^* , we may compare \hat{z} with z^* .
 - ▶ How to evaluate the performance of \hat{x} **without** solving the LP?
- ▶ If we can find an **upper bound** of z^* , that works!
 - ▶ z^* cannot be greater than the upper bound.
 - ▶ So if \hat{z} is close to the upper bound, \hat{x} is quite good.¹

¹You know 97 is quite high without knowing the highest in this class.

Upper bounds of a maximization LP

- ▶ How to find an upper bound of z^* for

$$\begin{aligned} z^* = \max \quad & 4x_1 + 5x_2 + 8x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq 6 \\ & 2x_1 + x_2 + 2x_3 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0? \end{aligned}$$

- ▶ How about this: Multiply the first constraint by 2, multiply the second constraint by 1, and then add them together:

$$\begin{aligned} 2(x_1 + 2x_2 + 3x_3) + (2x_1 + x_2 + 2x_3) &\leq 2 \times 6 + 4 \\ \Leftrightarrow 4x_1 + 5x_2 + 8x_3 &\leq 16. \end{aligned}$$

- ▶ Compare this with the objective function, we know $z^* \leq 16$.
 - ▶ Maybe z^* is exactly 16 (and the upper bound is **tight**). However, we do not know it here.
 - ▶ $\hat{z} = 15$ is close to $z^* = 16$, so \hat{x} is quite good.

Upper bounds of a maximization LP

- ▶ How to find an upper bound of z^* for this one?

$$\begin{aligned} z^* = \max \quad & 3x_1 + 4x_2 + 8x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq 6 \\ & 2x_1 + x_2 + 2x_3 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

- ▶ 16 is also an upper bound:

$$\begin{aligned} & 3x_1 + 4x_2 + 8x_3 \\ & \leq 4x_1 + 5x_2 + 8x_3 \quad (\text{because } x_1 \geq 0, x_2 \geq 0) \\ & = 2(x_1 + 2x_2 + 3x_3) + (2x_1 + x_2 + 2x_3) \\ & \leq 2 \times 6 + 4 = 16. \end{aligned}$$

- ▶ It is quite likely that 16 is not a tight upper bound and there is a better one. How to improve our upper bound?

Better upper bounds?

$$\begin{aligned}
 z^* = \max \quad & 3x_1 + 4x_2 + 8x_3 \\
 \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq 6 \\
 & 2x_1 + x_2 + 2x_3 \leq 4 \\
 & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.
 \end{aligned}$$

- ▶ Changing **coefficients** multiplied on the two constraints modifies the proposed upper bound.
 - ▶ Different coefficients result in different **linear combinations**.
- ▶ Let's call the two coefficients y_1 and y_2 , respectively:

$$\begin{array}{rcl}
 x_1 + & 2x_2 + & 3x_3 \leq 6 \quad (\times y_1) \\
 2x_1 + & x_2 + & 2x_3 \leq 4 \quad (\times y_2) \\
 \hline
 (y_1 + 2y_2)x_1 + & (2y_1 + y_2)x_2 + & (3y_1 + 2y_2)x_3 \leq 6y_1 + 4y_2
 \end{array}$$

- ▶ We need $y_1 \geq 0$ and $y_2 \geq 0$ to preserve the “ \leq ”.
- ▶ When do we have $z^* \leq 6y_1 + 4y_2$?

Looking for the lowest upper bound

- ▶ So we look for two variables y_1 and y_2 such that:
 - ▶ $y_1 \geq 0$ and $y_2 \geq 0$.
 - ▶ $3 \leq y_1 + 2y_2$, $4 \leq 2y_1 + y_2$, and $8 \leq 3y_1 + 2y_2$.
 - ▶ Then $z^* \leq 6y_1 + 4y_2$.
- ▶ To try our **best** to look for an upper bound, we minimize $6y_1 + 4y_2$. We are solving **another LP!**

$$\begin{array}{ll} \max & 3x_1 + 4x_2 + 8x_3 \\ \text{s.t.} & x_1 + 2x_2 + 3x_3 \leq 6 \\ & 2x_1 + x_2 + 2x_3 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{array} \quad \Rightarrow \quad \begin{array}{ll} \min & 6y_1 + 4y_2 \\ \text{s.t.} & y_1 + 2y_2 \geq 3 \\ & 2y_1 + y_2 \geq 4 \\ & 3y_1 + 2y_2 \geq 8 \\ & y_1 \geq 0, y_2 \geq 0. \end{array}$$

- ▶ We call the original LP the **primal** LP and the new one its **dual** LP.
- ▶ This idea applies to **any** LP. Let's see more examples.

Nonpositive or free variables

- ▶ Suppose variables are not all nonnegative:

$$\begin{aligned}
 z^* = \max \quad & 3x_1 + 4x_2 + 8x_3 \\
 \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq 6 \\
 & 2x_1 + x_2 + 2x_3 \leq 4 \\
 & x_1 \geq 0, \quad x_2 \leq 0, \quad x_3 \text{ urs.}
 \end{aligned}$$

- ▶ If we want

$$\leq (y_1 + 2y_2)x_1 + (2y_1 + y_2)x_2 + (3y_1 + 2y_2)x_3,$$

now we need

$$\begin{aligned}
 y_1 + 2y_2 &\geq 3 && \text{because } x_1 \geq 0, \\
 2y_1 + y_2 &\leq 4 && \text{because } x_2 \leq 0, \text{ and} \\
 3y_1 + 2y_2 &= 8 && \text{because } x_3 \text{ is free.}
 \end{aligned}$$

Nonpositive or free variables

- ▶ So the primal and dual LPs are

$$\begin{array}{llll} \max & 3x_1 & + & 4x_2 & + & 8x_3 \\ \text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 \\ & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\ & x_1 \geq 0, & x_2 \leq 0, & x_3 \text{ urs.} \end{array}$$

and

$$\begin{array}{llll} \min & 6y_1 & + & 4y_2 \\ \text{s.t.} & y_1 & + & 2y_2 & \geq & 3 \\ & 2y_1 & + & y_2 & \leq & 4 \\ & 3y_1 & + & 2y_2 & = & 8 \\ & y_1 \geq 0, & y_2 \geq 0. \end{array}$$

- ▶ Some observations:
 - ▶ Primal max \Rightarrow Dual min.
 - ▶ Primal objective \Rightarrow Dual RHS.
 - ▶ Primal RHS \Rightarrow Dual objective.
- ▶ Moreover:
 - ▶ Primal " ≥ 0 " variable \Rightarrow Dual " \geq " constraint.
 - ▶ Primal " ≤ 0 " variable \Rightarrow Dual " \leq " constraint.
 - ▶ Primal free variable \Rightarrow Dual " $=$ " constraint.
- ▶ What if we have " \geq " or " $=$ " primal constraints?

No-less-than and equality constraints

- ▶ Suppose constraints are not all “ \leq ”:

$$\begin{aligned} z^* = \max \quad & 3x_1 + 4x_2 + 8x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \geq 6 \\ & 2x_1 + x_2 + 2x_3 = 4 \\ & x_1 \geq 0, x_2 \leq 0, x_3 \text{ urs.} \end{aligned}$$

- ▶ To obtain

$$y_1(x_1 + 2x_2 + 3x_3) + y_2(2x_1 + x_2 + 2x_3) \leq 6y_1 + 4y_2,$$

we now need $y_1 \leq 0$. y_2 can be of any sign (i.e., free).

No-less-than and equality constraints

- So the primal and dual LPs are

$$\begin{array}{ll}
 \max & 3x_1 + 4x_2 + 8x_3 \\
 \text{s.t.} & x_1 + 2x_2 + 3x_3 \geq 6 \\
 & 2x_1 + x_2 + 2x_3 = 4 \\
 & x_1 \geq 0, x_2 \leq 0, x_3 \text{ urs.}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ll}
 \min & 6y_1 + 4y_2 \\
 \text{s.t.} & y_1 + 2y_2 \geq 3 \\
 & 2y_1 + y_2 \leq 4 \\
 & 3y_1 + 2y_2 = 8 \\
 & y_1 \leq 0, y_2 \text{ urs.}
 \end{array}$$

- Some more observations:
- Primal “ \leq ” constraint \Rightarrow Dual “ ≥ 0 ” variable.
 - Primal “ \geq ” constraint \Rightarrow Dual “ ≤ 0 ” variable.
 - Primal “ $=$ ” constraint \Rightarrow Dual free variable.

The general rule

- In general, if the primal LP is

$$\begin{array}{llllll} \max & c_1x_1 & + & c_2x_2 & + & c_3x_3 \\ \text{s.t.} & A_{11}x_1 & + & A_{12}x_2 & + & A_{13}x_3 & \geq & b_1 \\ & A_{21}x_1 & + & A_{22}x_2 & + & A_{23}x_3 & \leq & b_2 \\ & A_{31}x_1 & + & A_{32}x_2 & + & A_{33}x_3 & = & b_3 \\ & x_1 \geq 0, & x_2 \leq 0, & x_3 \text{ urs.}, & & & & \end{array}$$

its dual LP is

$$\begin{array}{llllll} \min & b_1y_1 & + & b_2y_2 & + & b_3y_3 \\ \text{s.t.} & A_{11}y_1 & + & A_{21}y_2 & + & A_{31}y_3 & \geq & c_1 \\ & A_{12}y_1 & + & A_{22}y_2 & + & A_{32}y_3 & \leq & c_2 \\ & A_{13}y_1 & + & A_{23}y_2 & + & A_{33}y_3 & = & c_3 \\ & y_1 \leq 0, & y_2 \geq 0, & y_3 \text{ urs.} & & & & \end{array}$$

- Note that the constraint coefficient matrix is “**transposed**”.

Matrix representation

- In general, if the primal LP

$$\begin{aligned} \max \quad & c_1x_1 + c_2x_2 + c_3x_3 \\ \text{s.t.} \quad & A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = b_1 \\ & A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = b_2 \\ & A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = b_3 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \end{aligned}$$

is in the **standard form**, its dual LP is

$$\begin{aligned} \min \quad & b_1y_1 + b_2y_2 + b_3y_3 \\ \text{s.t.} \quad & A_{11}y_1 + A_{21}y_2 + A_{31}y_3 \geq c_1 \\ & A_{12}y_1 + A_{22}y_2 + A_{32}y_3 \geq c_2 \\ & A_{13}y_1 + A_{23}y_2 + A_{33}y_3 \geq c_3. \end{aligned}$$

- In matrix representation:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

and

$$\begin{aligned} \min \quad & y^T b \\ \text{s.t.} \quad & y^T A \geq c^T. \end{aligned}$$

The dual LP for a minimization primal LP

- ▶ For a minimization LP, its dual LP is to **maximize** the **lower bound**.
- ▶ Rules for the directions of variables and constraints are **reversed**:

$$\begin{array}{ll}
 \min & 3x_1 + 4x_2 + 8x_3 \\
 \text{s.t.} & x_1 + 2x_2 + 3x_3 \geq 6 \\
 & 2x_1 + x_2 + 2x_3 \leq 4 \\
 & x_1 \geq 0, x_2 \leq 0, x_3 \text{ urs.}
 \end{array}
 \Rightarrow
 \begin{array}{ll}
 \max & 6y_1 + 4y_2 \\
 \text{s.t.} & y_1 + 2y_2 \leq 3 \\
 & 2y_1 + y_2 \geq 4 \\
 & 3y_1 + 2y_2 = 8 \\
 & y_1 \geq 0, y_2 \leq 0.
 \end{array}$$

- ▶ Note that

$$\begin{aligned}
 & 3x_1 + 4x_2 + 8x_3 \\
 \geq & (y_1 + 2y_2)x_1 + (2y_1 + y_2)x_2 + (3y_1 + 2y_2)x_3 \\
 \geq & (x_1 + 2x_2 + 3x_3)y_1 + (2x_1 + x_2 + 2x_3)y_2 \\
 \geq & 6y_1 + 4y_2.
 \end{aligned}$$

The general rule, uniqueness, and symmetry

- ▶ The general rule for finding the dual LP:

Obj. function	max	min	Obj. function
Constraint	\leq	≥ 0	Variable
	\geq	≤ 0	
	$=$	urs.	
Variable	≥ 0	\geq	Constraint
	≤ 0	\leq	
	urs.	$=$	

- ▶ If the primal LP is a maximization problem, do it from left to right.
- ▶ If the primal LP is a minimization problem, do it from right to left.

Proposition 1 (Uniqueness and symmetry of duality)

For any primal LP, there is a unique dual, whose dual is the primal.

Examples of primal-dual pairs

► Example 1:

$$\begin{array}{ll}
 \min & 2x_1 + 3x_2 \\
 \text{s.t.} & 4x_1 + x_2 \leq 9 \\
 & x_1 \geq 6 \\
 & 2x_1 - x_2 \geq 8 \\
 & x_1 \leq 0, x_2 \text{ urs.}
 \end{array}
 \Leftrightarrow
 \begin{array}{ll}
 \max & 9y_1 + 6y_2 + 8y_3 \\
 \text{s.t.} & 4y_1 + y_2 + 2y_3 \geq 2 \\
 & y_1 - y_3 = 3 \\
 & y_1 \leq 0, y_2 \geq 0, y_3 \geq 0.
 \end{array}$$

► Example 2:

$$\begin{array}{ll}
 \max & 3x_1 - x_2 \\
 \text{s.t.} & x_1 + 2x_2 = 6 \\
 & 3x_1 + 3x_2 \leq -4 \\
 & x_1 \text{ urs.}, x_2 \geq 0.
 \end{array}
 \Leftrightarrow
 \begin{array}{ll}
 \min & 6y_1 - 4y_2 \\
 \text{s.t.} & y_1 + 3y_2 = 3 \\
 & 2y_1 + 3y_2 \geq -1 \\
 & y_1 \text{ urs.}, y_2 \geq 0.
 \end{array}$$

Road map

- ▶ Primal-dual pairs.
- ▶ **Duality theorems.**
- ▶ Shadow prices.

Duality theorems

- ▶ Duality provides many interesting properties.
- ▶ We will illustrate these properties for standard form primal LPs:

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array} \quad (1)$$

- ▶ It can be shown that all the properties that we will introduce apply to other primal-dual pairs.

Weak duality

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array}$$

- ▶ The dual LP provides an **upper bound** of the primal LP.

Proposition 2 (Weak duality)

For the LPs defined in (1), if x and y are primal and dual feasible, then $c^T x \leq y^T b$.

Proof. As long as x and y are primal and dual feasible, we have

$$\begin{aligned} c^T x &\leq y^T Ax && (x \geq 0 \text{ and } y^T A \geq c^T) \\ &\leq y^T b && (Ax = b). \end{aligned}$$

Therefore, weak duality holds. □

Sufficiency of optimality

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array}$$

- ▶ We now have a **sufficient condition** for optimal solutions.

Proposition 3 (Sufficient condition for optimality)

If \bar{x} and \bar{y} are primal and dual feasible and $c^T \bar{x} = \bar{y}^T b$, then \bar{x} and \bar{y} are primal and dual optimal.

Proof. For all dual feasible y , we have $c^T \bar{x} \leq y^T b$ by weak duality. But we are given that $c^T \bar{x} = \bar{y}^T b$, so we have $\bar{y}^T b \leq y^T b$ for all dual feasible y . This just tells us that \bar{y} is dual optimal. For \bar{x} it is the same. \square

- ▶ Given a primal feasible solution \bar{x} , if we can find a dual feasible solution so that their objective values are **identical**, \bar{x} is optimal.

The dual optimal solution

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array}$$

- If we have solved the primal LP, the **dual optimal solution** is there.

Proposition 4 (Dual optimal solution)

For the LPs defined in (1), if \bar{x} is primal optimal with basis B , then $\bar{y}^T = c_B^T A_B^{-1}$ is dual optimal.

Proof. Because B is optimal, the reduced costs $c_B^T A_B^{-1} A_N - c_N^T \geq 0$. As $c_B^T = c_B^T A_B^{-1} A_B$, we have

$$\bar{y}^T A = c_B^T A_B^{-1} A = c_B^T A_B^{-1} [A_B \quad A_N] \geq [c_B^T \quad c_N^T] = c^T$$

and thus \bar{y} is dual feasible. As $\bar{y}^T b = c_B^T A_B^{-1} b = c_B^T x_B = c^T x$, \bar{x} and \bar{y} have the same objective value and are thus both optimal. \square

Strong duality

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array}$$

- The fact that $c_B^T A_B^{-1}$ is dual optimal implies **strong duality**:

Proposition 5 (Strong duality)

For the LPs defined in (1), \bar{x} and \bar{y} are primal and dual optimal if and only if \bar{x} and \bar{y} are primal and dual feasible and $c^T \bar{x} = \bar{y}^T b$.

Proof. To prove this if-and-only-if statement:

- (\Leftarrow): By Proposition 3.
- (\Rightarrow): As $c_B^T A_B^{-1}$ is an dual optimal solution, the dual optimal objective value is $c_B^T A_B^{-1} b$, which equals the primal optimal objective value $c^T \bar{x}$.
As \bar{y} is dual optimal, $\bar{y}^T b = c_B^T A_B^{-1} b = c^T \bar{x}$.² □

²As the dual LP may or may not have a unique optimal solution, \bar{y} and $c_B^T A_B^{-1}$ may or may not be identical. In either case, the statement holds.

Implications of strong duality

- ▶ Strong duality certainly implies weak duality.
 - ▶ Weak duality says that the dual LP provides a bound.
 - ▶ Strong duality says that the bound is **tight**, i.e., cannot be improved.
- ▶ The primal and dual LPs are **equivalent**.
- ▶ Given the result of one LP, we may predict the result of its dual:

Primal	Dual		
	Infeasible	Unbounded	Finitely optimal
Infeasible	✓	✓	×
Unbounded	✓	×	×
Finitely optimal	×	×	✓

- ▶ ✓ means possible, × means impossible.
- ▶ Primal unbounded \Rightarrow no upper bound \Rightarrow dual infeasible.
- ▶ Primal finitely optimal \Rightarrow finite objective value \Rightarrow dual finitely optimal.
- ▶ If primal is infeasible, the dual may still be infeasible (by examples).

Complementary slackness

- ▶ Consider w , the **slack** variables of the dual LP:

$$\begin{aligned} \min \quad & y^T b \\ \text{s.t.} \quad & y^T A - w^T = c^T \\ & w \geq 0. \end{aligned} \tag{2}$$

Proposition 6 (Complementary slackness)

For the primal defined in (1) and dual defined in (2), \bar{x} and (\bar{y}, \bar{w}) are primal and dual optimal if and only if $\bar{w}^T \bar{x} = 0$.

Proof. We have $c^T \bar{x} = (\bar{y}^T A - \bar{w}^T) \bar{x} = \bar{y}^T A \bar{x} - \bar{w}^T \bar{x} = \bar{y}^T b - \bar{w}^T \bar{x}$. Therefore, $\bar{w}^T \bar{x} = 0$ if and only if $c^T \bar{x} = \bar{y}^T b$, i.e., \bar{x} and (\bar{y}, \bar{w}) are primal and dual optimal according to strong duality. \square

- ▶ Note that $\bar{w}^T \bar{x} = 0$ if and only if $\bar{w}_i \bar{x}_i = 0$ for all i as $\bar{x} \geq 0$ and $\bar{w} \geq 0$.
- ▶ If a dual (respectively, primal) constraint is **nonbinding**, the corresponding primal (respectively, dual) variable is **zero**.

Why duality?

- ▶ Why duality? Given an LP:
 - ▶ We may solve it directly.
 - ▶ Or we may solve the dual LP and then get the primal optimal solution (by Proposition 4).
- ▶ Why bothering?
- ▶ Recall that the computation time of the simplex method is roughly proportional to m^3 .
 - ▶ m is the number of functional constraints of the original LP.
 - ▶ And n , the number of variables of the original LP, does not matter a lot.
- ▶ If $m \gg n$, solving the dual LP can take a significantly **shorter time** than solving the primal!
- ▶ There are many other benefits for having duality. We will see some more in this course.
- ▶ Read Sections 6.1, 6.3, and 6.4 carefully.

Road map

- ▶ Primal-dual pairs.
- ▶ Duality theorems.
- ▶ **Shadow prices.**

A product mix problem

- ▶ Suppose we produce tables and chairs with wood and labors. In total we have six units of wood and six labor hours.
 - ▶ Each table is sold at \$3 and requires 2 units of wood and 1 labor hour.
 - ▶ Each chair is sold at \$1 and requires 1 unit of wood and 2 labor hours.

How may we formulate an LP to maximize our sales revenue?

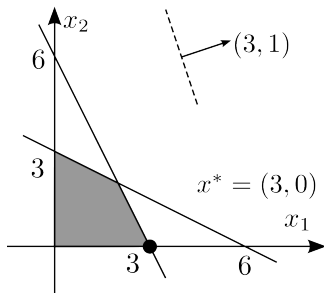
- ▶ The formulation is

x_1 = number of tables produced

x_2 = number of chairs produced.

$$\begin{array}{ll} \max & 3x_1 + x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 6 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{array}$$

- ▶ The optimal solution is $x^* = (3, 0)$.



“What-if” questions

- ▶ In practice, people often ask “**what-if**” questions:
 - ▶ What if the unit price of chairs becomes \$2?
 - ▶ What if each table requires 3 unit of wood?
 - ▶ What if we have 10 units of wood?
- ▶ Why what-if questions?
 - ▶ Parameters may fluctuate.
 - ▶ Estimation of parameters may be inaccurate.
 - ▶ Looking for ways to improve the business.
- ▶ For realistic problems, what-if questions can be hard.
 - ▶ Even though it may be just a tiny modification of one parameter, the optimal solution may change a lot.
- ▶ The tool for answering what-if questions is **sensitivity analysis**.

Humboldt Redwood



- ▶ Pacific Lumber Company (now Humboldt Redwood) has over 200,000 acres of forests and five mills in Humboldt County.
- ▶ **Sustainability** is important in making operational decisions.
- ▶ They contracted with an OR team to develop a 120-year forest ecosystem management plan.
 - ▶ The LP optimizes the timberland operations for maximizing profitability while satisfying constraints including sustainability.
 - ▶ The model has around 8,500 functional constraints and 353,000 variables.
- ▶ The environment keeps **changing!**
 - ▶ E.g., climate, supply and demand, logging costs, and regulations.
 - ▶ Sensitivity analysis is applied.
- ▶ Read the application vignette in Section 6.7 and the article on CEIBA.

“What-if” questions

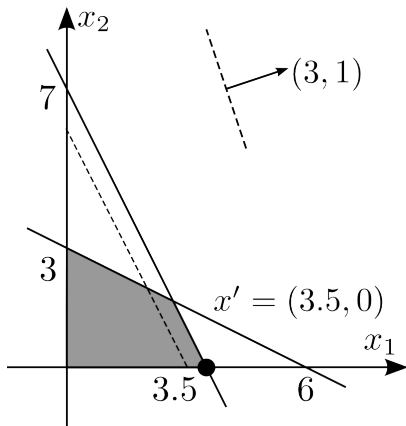
- ▶ In general, what-if questions can always be answered by formulating and solving a new optimization problem **from scratch**.
- ▶ But this may be too time consuming!
- ▶ By sensitivity analysis techniques:
 - ▶ The original optimal tableau provides useful information.
 - ▶ We typically start from the original optimal bfs and do **just a few** iterations to reach the new optimal bfs.
 - ▶ Duality provides a theoretical background.
- ▶ Here we want to introduce just one type of what-if question: What if I have **additional** units of a certain **resource**?
- ▶ Consider the following scenario:
 - ▶ One day, a salesperson enters your office and wants to offer you one additional unit of wood at \$1. Should you accept or reject?

One more unit of wood

- ▶ To answer this question, you may formulate a new LP:

$$\begin{array}{llll}
 \max & 3x_1 & + & x_2 \\
 \text{s.t.} & 2x_1 & + & x_2 \leq 7 \\
 & x_1 & + & 2x_2 \leq 6 \\
 & x_i & \geq & 0 \quad \forall i = 1, 2.
 \end{array}$$

- ▶ The new objective value $z' = 3 \times 3.5 = 10.5$ is larger than the old objective value $z^* = 9$.
- ▶ It is good to accept the offer (at the unit price \$1).
 - ▶ We earn \$0.5 as our **net benefit**.

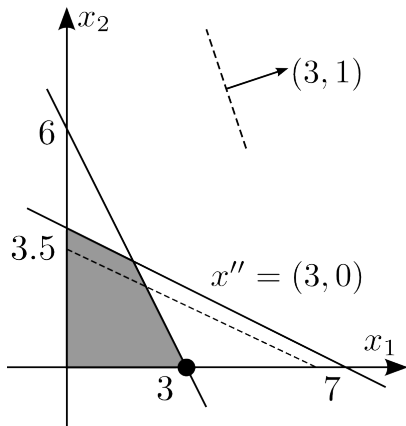


One more labor hour

- ▶ Suppose instead of offering one additional unit of wood, the salesperson offers one additional labor hour at \$1.

$$\begin{array}{llll} \max & 3x_1 & + & x_2 \\ \text{s.t.} & 2x_1 & + & x_2 \leq 6 \\ & x_1 & + & 2x_2 \leq 7 \\ & x_i & \geq & 0 \quad \forall i = 1, 2. \end{array}$$

- ▶ The new objective value is **the same as** the old objective value.
- ▶ It is not worthwhile to buy it: The objective value does not increase.
 - ▶ The **net loss** is \$1.



Shadow prices

- ▶ For each resource, there is a **maximum amount of price** we are willing to pay for one additional unit.
 - ▶ That depends on the net benefit of that one additional unit.
 - ▶ For wood, this price is \$1.5. For labor hours, this price is \$0.
- ▶ This motivates us to define **shadow prices** for each constraint:

Definition 1 (Shadow price)

For an LP that has an optimal solution, the shadow price of a constraint is the amount of objective value increased when the RHS of that constraint is increased by 1, assuming the current optimal basis remains optimal.

- ▶ So for our table-chair example, the shadow prices for constraints 1 and 2 are 1.5 and 0, respectively.
- ▶ For shadow prices, see Section 4.7.
- ▶ Note that we **assume** that the current optimal basis does not change.

Assuming the optimal basis does not change

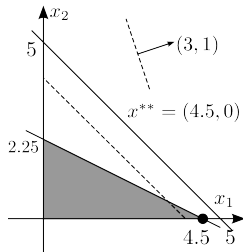
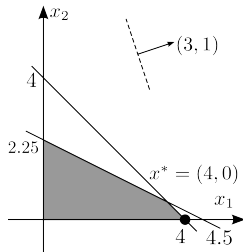
- ▶ Consider another example:

$$\begin{aligned} z^* = \max \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 4.5 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$

- ▶ If we want to find the shadow price of constraint 1, we may try to solve a new LP:

$$\begin{aligned} z^{**} = \max \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & x_1 + 2x_2 \leq 4.5 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$

- ▶ Though $z^{**} = 13.5$ and $z^* = 12$, the shadow price is $15 - 12 = 3$, **not** 1.5!
- ▶ Shadow prices measure the **rate** of improvement.



Signs of shadow prices

- As a shadow price measures how the objective value is **increased**, its sign is determined based on how the feasible region changes:

Proposition 7 (Signs of shadow prices)

For any LP, the sign of a shadow price follows the rule below:

<i>Objective function</i>	<i>Constraint</i>		
	\leq	\geq	$=$
max	≥ 0	≤ 0	Free
min	≤ 0	≥ 0	Free

Nonbinding constraints' shadow prices

- ▶ If shifting a constraint does not affect the optimal solution, the shadow price must be **zero**.³

Proposition 8

Shadow prices are zero for constraints that are nonbinding at the optimal solution.

- ▶ Now we know finding shadow prices allows us to answer the questions regarding additional units of resources.
- ▶ But how to find **all** shadow prices?
 - ▶ Let m be the number of constraints.
 - ▶ Is there a better way than solving m LPs?
 - ▶ Duality helps!

³Not all binding constraints has nonzero shadow prices. Why?

Dual optimal solution provide shadow prices

Proposition 9

For any LP, shadow prices equal the values of dual variables in the dual optimal solution.

Proof. Let B be the old optimal basis and $z = c_B^T A_B^{-1} b$ be the old objective value. If b_1 becomes $b'_1 = b_1 + 1$, then z becomes

$$z' = c_B^T A_B^{-1} \left(b + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) = z + (c_B^T A_B^{-1})_1.$$

So the shadow price of constraint 1 is $(c_B^T A_B^{-1})_1$. In general, the shadow price of constraint i is $(c_B^T A_B^{-1})_i$. As $c_B^T A_B^{-1}$ is the dual optimal solution, the proof is complete. □

An example

- ▶ What are the shadow prices?

$$\begin{array}{ll}
 \min & 6x_1 + 4x_2 \\
 \text{s.t.} & x_1 + x_2 \geq 2 \\
 & 3x_1 + x_2 \geq 1 \\
 & x_i \geq 0 \quad \forall i = 1, 2.
 \end{array}$$

- ▶ We solve the dual LP

$$\begin{array}{ll}
 \max & 2y_1 + y_2 \\
 \text{s.t.} & y_1 + 3y_2 \leq 6 \\
 & y_1 + y_2 \leq 4 \\
 & y_i \geq 0 \quad \forall i = 1, 2.
 \end{array}$$

The dual optimal solution is $y^* = (4, 0)$.

- ▶ So shadow prices are 4 and 0, respectively.

