Operations Research, Spring 2016
Final Exam

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June 2, 2016

Note. You do not need to return these problem sheets; write down all your answers on the answer sheets. In total there are 130 points. Get as many as possible!

1. (15 points) Apply the two phase method with the smallest index rule to solve the following LP. Write down all the iterations to get full credits.

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 6 \\ & 2x_1 + x_2 \geq 6 \\ & x_1 \leq 4 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$

2. (15 points) Consider the following LP

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 + x_3 + x_4 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 \leq 10 \\ & 2x_1 + x_2 + 2x_3 + x_4 \leq 12 \\ & x_i \geq 0 \quad \forall i = 1, ..., 4. \end{aligned}$$

   (a) (5 points) Find the dual LP.

   (b) (10 points) Find the shadow prices of the constraints.

3. (15 points) Consider the following IP:

$$\begin{aligned} \max \quad & 5x_1 + 2x_2 + 4x_3 + 5x_4 \\ \text{s.t.} \quad & 2x_1 + 2x_2 + x_3 + 3x_4 \leq 10 \\ & 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, 0 \leq x_3 \leq 3, 2 \leq x_4 \leq 4 \\ & x_i \in \mathbb{Z} \quad \forall i = 1, ..., 4. \end{aligned}$$

   Apply the branch-and-bound algorithm to solve this problem. Write down the complete branching tree to get full credits. You are free to choose the node and variable to branch on.

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4. (20 points; 10 points each) A city government is going to offer a shuttle service from its main train station to a hospital. The service will run from 9 am to 9 pm for each day. Due to limited budgets, at most 5 shuttles may be scheduled, and these shuttles must be scheduled at 10 am, 11 am, ..., 9 pm. Among these 12 time points, there must be a shuttle at 9 pm so that customers arriving between 8 pm and 9 pm get served. Therefore, the question is really to allocate 4 shuttles to the remaining 11 candidate time points. Let the 12 hours (9 am to 10 am, 10 am to 11 am, ..., and 8 pm to 9 pm) be labeled as hour 1, 2, ..., and 12, respectively. The number of customers who will use this service in hour i is estimated to be D_i. For simplicity, let's assume that customers arriving in hour i all arrive at the end of hour i. For example, customers arriving in hour 1 all arrive exactly at 10 am. Then if there is a shuttle at 10 am, their waiting times are all 0. If instead there is a shuttle at 11 am, their writing times are all 1 hour. A shuttle can take at most K customers. The government wants to minimize the total waiting time of these customers.

   (a) Suppose that $K > \sum_{i=1}^{12} D_i$, formulate the government's problem as a linear integer program.

   (b) Suppose that $K < \sum_{i=1}^{12} D_i$, $5K > \sum_{i=1}^{12} D_i$, and $D_i < K$ for all $i = 1, ..., 12$, formulate the government's problem as a linear integer program.

5. (20 points) A retailer is going to purchase a product from a manufacturer. If the retailer obtains q units from the manufacturer, she will sell all of them to the market at the unit price $a - q$. The retailer now needs to propose a lump-sum fee t to the manufacturer for purchasing q units. Note that t is the total amount paid to the manufacturer, not a price for a unit. The retailer's objective is to maximize her total profit. Nevertheless, there are two constraints. First, the manufacturer can produce at most b units due to limited capacity. Second, the price must be high enough so that the manufacturer can make a nonnegative profit by accepting this offer. By signing the contract (to deliver q units in exchange of t dollars), the manufacturer's profit is $t - cq$, where c is the unit product cost. It is known that $b < a$ and $c < a$. Let's assume that q is continuous and nonnegative.

   (a) (5 points) Formulate the retailer's problem as a nonlinear program.

   (b) (10 points) Solve the retailer's problem to obtain her optimal order quantity q^* and lump-sum fee t^* as functions of a, b, and c.

   (c) (5 points) Mathematically show how a, b, and c affects the retailer's optimal order quantity q^*. Intuitively explain why.

6. (15 points; 5 points each) Consider the following NLP:

$$\min_{x \in \mathbb{R}^2} \ (x_1 - 4)^2 + (x_2 - 2)^2 + x_1 x_2.$$

   (a) Starting at $(0, 0)$ and run one iteration of gradient descent to get to the next solution.

   (b) Starting at $(2, 2)$ and run one iteration of gradient descent to get to the next solution.

   (c) Starting at $(2, 2)$ and run one iteration of Newton's method to get to the next solution.

7. (10 points; 5 points each) Consider the following LP

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq 6 \\ & 2x_1 + x_2 + 2x_3 \leq 16 \\ & x_i \geq 0 \quad \forall i = 1, 2, 3, \end{aligned}$$

whose optimal tableau (for the standard form) is

$$\begin{array}{ccccc|c} 0 & 1 & 2 & 1 & 0 & 6 \\ \hline 1 & 2 & 3 & 1 & 0 & x_1 = 6 \\ 0 & -3 & -4 & -2 & 1 & x_5 = 4 \end{array}$$

   (a) Find a range of Δ such that the optimal basis does not change when the objective function becomes max $x_1 + (1 + \Delta)x_2 + x_3$.

   (b) Find a range of Δ such that the optimal basis does not change when the objective function becomes max $(1 + \Delta)x_1 + x_2 + x_3$.

8. (10 points; 5 points each) Consider the following zero-sum game, where the numbers are the payoffs of the row player:

	L	C	R
T	4	−4	10
M	2	3	1
B	−6	5	7

   (a) Prove that there is no pure-strategy Nash equilibrium without completely testing all action profiles.

   (b) Show that the two players' optimization problems for finding the optimal mixed strategies are a pair of primal and dual LPs.

9. (10 points) Consider a knapsack problem

$$\begin{aligned} \max \quad & 4x_1 + 3x_2 + 2x_3 + 2x_4 \\ \text{s.t.} \quad & 5x_1 + 4x_2 + 3x_3 + 2x_4 \leq 10 \\ & x_i \in \{0, 1\} \quad \forall i = 1, ..., 4. \end{aligned}$$

To apply the genetic algorithm to solve the knapsack problem, we may define a chromosome as $(x_1, x_2, ..., x_n)$, where $x_i = 1$ if item i is selected and 0 otherwise. The fitness of a chromosome is the sum of values of selected items times 100 minus the sum of weights of selected items if it is feasible or -100 otherwise. Let the initial population be the set of chromosomes $\{(x_1, x_2, ..., x_n) | x_1 + \cdots + x_n = 1\}$.

(a) (2 points) To run one iteration of genetic algorithm, find the most fitted two chromosomes that can be used to generate the next two chromosomes.

(b) (4 points) Suppose that the two parent chromosomes do not mutate, and the crossover point is between the first and second variables, find the two child chromosomes. Then update the pool of chromosomes by putting the most fitted four chromosomes among the six ones into the pool.

(c) (4 points) From the pool of chromosomes obtained in Part (b), find the least fitted two chromosomes, crossover between the second and third variables, and then update the pool of chromosomes in the same way.