Final

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. Let $x_1, x_2, \cdots, x_{2n-1}, x_{2n}$ be a sequence of $2n$ real numbers. Design an algorithm to partition the numbers into n pairs such that the maximum of the n sums of pair is minimized. It may be intuitively easy to get a correct solution. You must explain how the algorithm can be designed using induction.

2. Compute the *next* table as in the KMP algorithm for the string $B[1..11] = bbabbabbaab$. Please show how $next[10]$ and $next[11]$ are computed from using preceding entries of the table.

3. Below is an algorithm skeleton for depth-first search utilizing a stack; assume that the input graph is undirected and connected. Modify the algorithm so that it prints out (the edges of) a DFS tree of the input graph. You should try not to change the overall structure of the original algorithm.

   Algorithm Simple_Nonrecursive_DFS (G, v);
   begin
       push v to *Stack*;
       while *Stack* is not empty **do**
           pop vertex w from *Stack*;
           if w is unmarked **then**
               mark w;
               for all edges (w, x) such that x is unmarked **do**
                   push x to *Stack*
   end

4. Design an algorithm for determining whether a given *acyclic* directed graph $G = (V, E)$ contains a directed Hamiltonian path. (Note: A directed *Hamiltonian path*

of a directed graph is a simple directed path that includes all vertices of the graph. An acyclic directed graph is one without directed cycles.) Please present your algorithm in an adequate pseudo code and make assumptions wherever necessary. The more efficient your algorithm is, the more points you will be credited for this problem. Explain why your algorithm is correct and give an analysis of its time complexity. What does the existence of a directed Hamiltonian path imply (when the directed graph is meant to model the dependency among tasks)? (15 points)

5. Consider an $m \times n$ matrix of non-negative integers. A path through the matrix consists of a sequence of cells of the matrix, starting anywhere in the first column (Column 1) and terminating in the last column (Column n). Two consecutive cells in a path constitute a step of the path and should go from one column to the next column and either remain on the same row or go up or down one row. The weight of a path is the sum of the integers in the cells along the path. Please try to reduce the problem to a shortest path problem in graphs. You should clearly identify the target problem and describe the conversion of one input to the other. (15 points)

6. What is wrong with the following algorithm for computing the minimum-cost spanning tree of a given weighted undirected graph (assumed to be connected)?

   > If the input is just a single-node graph, return the single node. Otherwise, divide the graph into two subgraphs and recursively compute their minimum-cost spanning trees. Finally, connect the two trees with an edge between the two subgraphs that has the minimum weight.

7. Let $G = (V, E)$ be a directed graph, and let T be a DFS tree of G. Prove that the intersection of the edges of T with the edges of any strongly connected component of G form a subtree of T.

8. In the proof (discussed in class) of the NP-hardness of the clique problem by reduction from the SAT problem, we convert an arbitrary boolean expression in CNF (input of the SAT problem) to an input graph of the clique problem. Please demonstrate the conversion by drawing the graph for the following boolean expression: $(w + x + \overline{y} + z) \cdot (\overline{x} + \overline{y}) \cdot (\overline{w} + x + z)$. Explain how you have drawn the graph systematically.

9. Solve one of the following two problems. (Note: If you try to solve both problems, I will randomly pick one of them to grade.)

   (a) The hitting set problem is as follows.

Given a collection C of subsets of a set S and a positive integer k, does S contain a hitting set for C of size k or smaller, that is, a subset $S' \subseteq S$ with $|S'| \leq k$ such that S' contains at least one element from each subset in C?

Prove that the hitting set problem is NP-complete.

(b) A variant of the Hamiltonian path problem is as follows.

Given an undirected graph $G = (V, E)$ and $u, v \in V$, does G have a Hamiltonian path from u to v? (A Hamiltonian path in a graph is a simple path that contains each vertex exactly once.)

Prove that this variant of the Hamiltonian path problem is NP-complete.

Appendix

- The vertex cover problem: given an undirected graph $G = (V, E)$ and an integer k, determine whether G has a vertex cover containing $\leq k$ vertices. (A *vertex cover* of G is a set of vertices such that every edge in G is incident to at least one of these vertices.)
  The vertex cover problem is complete.

- The Hamiltonian cycle problem: given an undirected graph G, does G have a Hamiltonian cycle? (A Hamiltonian cycle in a graph is a cycle that contains each vertex, except the starting vertex of the cycle, exactly once.)
  The problem is NP-complete.