Midterm

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. Given any binary tree T, let l_T denote the number of its leaves and m_T the number of its internal nodes.

   (a) Prove *by induction* that, if every internal node of T has two children, then $l_T - m_T = 1$. (8 points)

   (b) Use the preceding result to show that, if T is a complete binary tree, then either $l_T - m_T = 1$ or $l_T - m_T = 0$. (2 points)

2. Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. Prove *by induction* that $(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 2^n$. (Hint: In the inductive step, try introducing a new variable that replaces two chosen numbers from the sequence.)

3. For each pair f,g of functions, indicate whether $f(n) = O(g(n))$ and/or $f(n) = \Omega(g(n))$. (Stirling's approximation: $n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + O(1/n))$.)

$f(n)$	$g(n)$
$2n + \log n$	$n + (\log n)^2$
$(\log n)^{\log n}$	n
3^n	$3^{\frac{n}{2}}$
$\log(n!)$	$\log(n^n)$

4. Modify the following code for determining the sum of the maximum consecutive subsequence so that it also records the start and end indices of the subsequence.

Algorithm Max_Consec_Subseq (X, n);
begin
  Global_Max $:= 0$;
  Suffix_Max $:= 0$;

for $i := 1$ **to** n **do**

  if $x[i] + \mathit{Suffix\_Max} > \mathit{Global\_Max}$ **then**

    $\mathit{Suffix\_Max} := \mathit{Suffix\_Max} + x[i]$;

    $\mathit{Global\_Max} := \mathit{Suffix\_Max}$

  else if $x[i] + \mathit{Suffix\_Max} > 0$ **then**

      $\mathit{Suffix\_Max} := \mathit{Suffix\_Max} + x[i]$

  else $\mathit{Suffix\_Max} := 0$

end

5. In a history exam problem, the students are asked to put several historical events into chronological order. Students who order all events correctly will receive full credit. Partial credits are awarded according to the longest (not necessarily contiguous) subsequence of events that are in the correct order relative to each other. Your task is to design an algorithm that determines the length of such a subsequence for the answer given by a student. Assume you already have a procedure that can find the longest *increasing* subsequence of a given sequence of distinct integers. Utilize the assumed procedure to obtain the needed algorithm.

6. Show all intermediate and the final AVL trees formed by inserting the numbers 4, 3, 0, 2, 1, 8, 5, 7, 9, and 6 (in this order). Please use the following ordering convention: the key of an internal node is larger than that of its left child and smaller than that of its right child. If a rotation is performed during an insertion, please also show the tree before the rotation. (15 points)

7. Apply the quicksort algorithm to the following array. Show the contents of the array after each partition operation. Please briefly describe your partition algorithm if it is different from the one we discussed in class.

7	1	5	11	2	10	9	3	6	12	4	8

8. We have discussed in class how to rearrange an array into a (max) heap using a bottom-up approach. Please present the approach in pseudocode. (15 points)

9. Draw a Huffman tree for a text with the following frequency distribution: $A : 12$, $B : 7$, $C : 3$, $D : 5$, $E : 15$, $F : 4$, $G : 1$, and $H : 2$.