Midterm

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. Given a set of $n+1$ numbers out of the first $2n$ (starting from 1) natural numbers 1, 2, 3, …, $2n$, prove *by induction* that there are two numbers in the set, one of which divides the other.

2. Construct a gray code of length $\lceil \log_2 14 \rceil$ $(= 4)$ for 14 objects. Show how the gray code is constructed *systematically* from gray codes of smaller lengths.

3. Let $T(h)$ denote the number of nodes in a smallest AVL tree of height h (smallest in the sense of having the least number of nodes); the height of an empty tree is defined to be 0.

   (a) Define a recurrence relation for $T(h)$ $(h \geq 0)$. Be sure to cover the base cases (or marginal cases).

   (b) Derive, based on the preceding recurrence relation, a lower bound for $T(h)$, showing that $T(h)$ grows at least exponentially with h. How do you infer, from the lower bound, the time complexity of performing a search operation on an AVL tree of size n?

4. Show all intermediate and the final AVL trees formed by inserting the numbers 8, 9, 7, 4, 6, 5, 1, 3, and 2 (in this order) into an empty tree. Please use the following ordering convention: the key of an internal node is larger than that of its left child and smaller than that of its right child. If a rotation is performed during an insertion, please also show the tree before the rotation. (15 points)

5. Design an algorithm that solves the following variant of the towers of Hanoi problem (adapted from Exercise 5.24 of Manber's book): Like in the original problem, there are three pegs, each capable of holding up to n disks (for some given n). Initially, the n disks (of different sizes) are arbitrarily distributed among the three pegs, all

in a decreasing order of sizes (from bottom to top). The goal is to move all the n disks, one at a time using only the pegs as temporary storage, to one of the three pegs, without violating the ordering constraint and with as few moves as possible. Please present your algorithm in an adequate pseudo code and make assumptions wherever necessary. (15 points)

6. Design an efficient algorithm that, given an array A of n integers and an integer x, determine whether A contains two integers whose sum is exactly x. Please present your algorithm in an adequate pseudo code and make assumptions wherever necessary. Give an analysis of its time complexity. The more efficient your algorithm is, the more points you will be credited for this problem.

7. Rearrange the following array into a (max) heap using the bottom-up approach.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	8	3	5	9	14	7	6	1	4	10	13	15	12	11

Show the result after each element is added to the part of array that already satisfies the heap property.

8. Draw a Huffman tree for a text with the following frequency distribution: $A : 14$, $B : 8$, $C : 7$, $D : 4$, $E : 18$, $F : 4$, $G : 3$, and $H : 2$.

9. Solve one of the following two problems:

   (a) Suppose that you are given an algorithm as a *black box* (you cannot see how it is designed) that has the following properties: If you input any sequence of real numbers and an integer k, the algorithm will answer "yes" or "no," indicating whether there is a subset of the numbers whose sum is exactly k. Show how to use this black box to find the subset whose sum is k, if it exists. You should use the black box $O(n)$ times (where n is the size of the sequence).

   (b) Prove that the sum of the heights of all nodes in a complete binary tree with n nodes is at most $n - 1$. (A complete binary tree with n nodes is one that can be compactly represented by an array A of size n, where the root is stored in $A[1]$ and the left and the right children of $A[i]$, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, are stored respectively in $A[2i]$ and $A[2i+1]$. Notice that, in Manber's book a complete binary tree is referred to as a balanced binary tree and a full binary tree as a complete binary tree. Manber's definitions seem to be less frequently used. Do not let the different names confuse you.)