String Processing
(Based on [Manber 1989])

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Data Compression

Problem

Given a text (a sequence of characters), find an encoding for the characters that satisfies the prefix constraint and that minimizes the total number of bits needed to encode the text.

The *prefix constraint* states that the prefixes of an encoding of one character must not be equal to a complete encoding of another character.

Denote the characters by $c_1, c_2, \cdots, c_n$ and their frequencies by $f_1, f_2, \cdots, f_n$. Given an encoding E in which a bit string s_i represents c_i, the length (number of bits) of the text encoded by using E is $\sum_{i=1}^{n} |s_i| \cdot f_i$.

A Code Tree

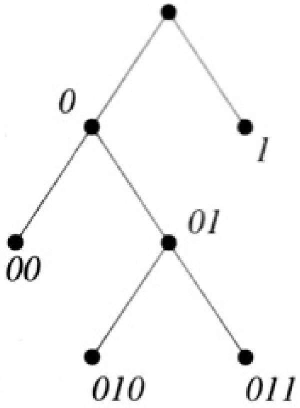

Figure 6.17 The tree representation of encoding.

Source: [Manber 1989].

A Huffman Tree

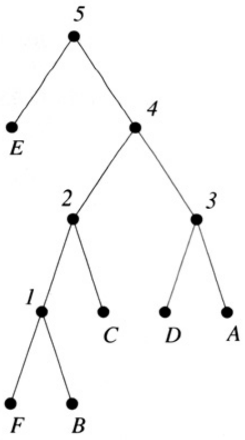

Figure 6.19 The Huffman tree for example 6.1.

Source: [Manber 1989]. (Frequencies: A: 5, B: 2, C: 3, D: 4, E: 10, F:1)

Huffman Encoding

Algorithm Huffman_Encoding (S, f);
  insert all characters into a heap H
    according to their frequencies;
  while H not empty **do**
    if *H contains only one character X* **then**
      make X the root of T
    else
      delete X and Y with lowest frequencies;
        from H;
      create Z with a frequency equal to the
        sum of the frequencies of X and Y;
      insert Z into H;
      make X and Y children of Z in T

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      make X *and* Y *children of* Z *in* T

What is its time complexity?

String Matching

IM NTU

Problem

Given two strings A $(= a_1 a_2 \cdots a_n)$ and B $(= b_1 b_2 \cdots b_m)$, find the first occurrence (if any) of B in A. In other words, find the smallest k such that, for all i, $1 \leq i \leq m$, we have $a_{k-1+i} = b_i$.

A *substring* of a string A is a consecutive sequence of characters $a_i a_{i+1} \cdots a_j$ from A.

Straightforward String Matching

IM NTU

$A = xyxxyxyxyyxyxyxyyxyxyxx$. $B = xyxyyxyxyxx$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
	x	y	x	x	y	x	y	x	y	y	x	y	x	y	x	y	y	x	y	x	y	x	x
1:	x	y	x	y	·	·	·																
2:		x	·	·	·																		
3:			x	y	·	·	·																
4:				x	y	x	y	y	·	·	·												
5:					x	·	·	·															
6:						x	y	x	y	y	x	y	x	y	x	x							
7:							x	·	·	·													
8:								x	y	x	·	·	·										
9:									x	·	·	·											
10:										x	·	·	·										
11:											x	y	x	y	y	·	·	·					
12:												x	·	·	·								
13:													x	y	x	y	y	x	y	x	y	x	x

Figure 6.20 An example of a straightforward string matching.

Source: [Manber 1989].

Straightforward String Matching (cont.)

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Straightforward String Matching (cont.)

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  - B $(= b_1 b_2 \cdots b_m)$ may be compared against
    - $a_1 a_2 \cdots a_m$,
    - $a_2 a_3 \cdots a_{m+1}$,
    - $\ldots$, and
    - $a_{n-m+1} a_{n-m+2} \cdots a_n$
  - For example, $A = xxxx \ldots xxxy$ and $B = xxxy$.

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  - For example, $A = xxxx \ldots xxxy$ and $B = xxxy$.
- So, the time complexity is $O(m \times n)$.
- We will exam the cause of defficiency.
- We then study an efficient algorithm, which is linear-time with a preprocessing stage.

Matching Against Itself

$B =$	x	y	x	y	y	x	y	x	y	x	x
		x	·	·	·						
			x	y	x	·	·	·			
				x	·	·	·				
					x	·	·	·			
						x	y	x	y	y	
							x	·	·	·	
								x	y	x	

Figure 6.21 Matching the pattern against itself.

Source: [Manber 1989].

The Values of *next*

$i =$	1	2	3	4	5	6	7	8	9	10	11
$B =$	x	y	x	y	y	x	y	x	y	x	x
$next =$	-1	0	0	1	2	0	1	2	3	4	3

Figure 6.22 The values of *next*.

Source: [Manber 1989].

The value of $next[j]$ tells the length of the longest proper prefix that is equal to a suffix of $b_1 b_2 \ldots b_j$.

The KMP Algorithm

Algorithm String_Match (A, n, B, m);
begin
    $j := 1;\ \ i := 1;$
    $Start := 0;$
    while $Start = 0$ and $i \leq n$ **do**
        if $B[j] = A[i]$ **then**
            $j := j + 1;\ \ i := i + 1$
        else
            $j := next[j] + 1;$
            if $j = 0$ **then**
                $j := 1;\ \ i := i + 1;$
        if $j = m + 1$ **then** $Start := i - m$
end

The KMP Algorithm (cont.)

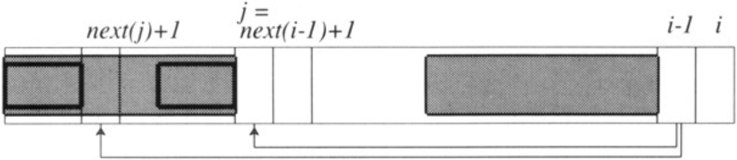

Figure 6.24 Computing next(i).

Source: [Manber 1989].

The KMP Algorithm (cont.)

Algorithm Compute_Next (B, m);
begin
    $next[1] := -1$; $next[2] := 0$;
    for $i := 3$ **to** m **do**
        $j := next[i-1] + 1$;
        while $B[i-1] \neq B[j]$ and $j > 0$ **do**
            $j := next[j] + 1$;
        $next[i] := j$
end

The KMP Algorithm (cont.)

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  - Because of backtracking, a_i may be compared against
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    - b_{j-1},
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The KMP Algorithm (cont.)

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  - Because of backtracking, a_i may be compared against
    - b_j,
    - b_{j-1},
    - $\ldots$, and
    - b_2
  - However, for these to happen, each of $a_{i-j+2}, a_{i-j+3}, \ldots, a_{i-1}$ was compared against the corresponding character in $b_1 b_2 \ldots b_{j-1}$ just once.

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  - We may re-assign the costs of comparing a_i against $b_{j-1}, b_{j-2}, \ldots, b_2$ to those of comparing $a_{i-j+2} a_{i-j+3} \ldots a_{i-1}$ against $b_1 b_2 \ldots b_{j-1}$.

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- Every a_i is incurred the cost of at most two comparisons.
- So, the time complexity is $O(n)$.

String Editing

Problem

Given two strings A $(= a_1 a_2 \cdots a_n)$ and B $(= b_1 b_2 \cdots b_m)$, find the minimum number of changes required to change A character by character such that it becomes equal to B.

Three types of changes (or edit steps) allowed: (1) insert, (2) delete, and (3) replace.

String Editing (cont.)

Let $C(i,j)$ denote the minimum cost of changing $A(i)$ to $B(j)$, where $A(i) = a_1 a_2 \cdots a_i$ and $B(j) = b_1 b_2 \cdots b_j$.

$$C(i,j) = \min \begin{cases} C(i-1,j)+1 & (\text{deleting } a_i) \\ C(i,j-1)+1 & (\text{inserting } b_j) \\ C(i-1,j-1)+1 & (a_i \rightarrow b_j) \\ C(i-1,j-1) & (a_i = b_j) \end{cases}$$

String Editing (cont.)

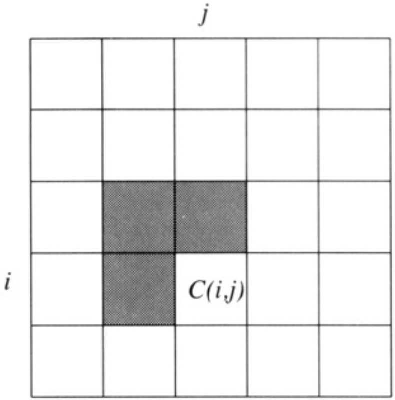

Figure 6.26 The dependencies of $C(i, j)$.

Source: [Manber 1989].

String Editing (cont.)

```
Algorithm Minimum_Edit_Distance (A, n, B, m);
    for i := 0 to n do C[i, 0] := i;
    for j := 1 to m do C[0, j] := j;
    for i := 1 to n do
        for j := 1 to m do
            x := C[i − 1, j] + 1;
            y := C[i, j − 1] + 1;
            if a_i = b_j then
                z := C[i − 1, j − 1]
            else
                z := C[i − 1, j − 1] + 1;
        C[i, j] := min(x, y, z)
```