Algorithms 2018: Basic Graph Algorithms

(Based on [Manber 1989])

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1 Introduction

The Königsberg Bridges Problem

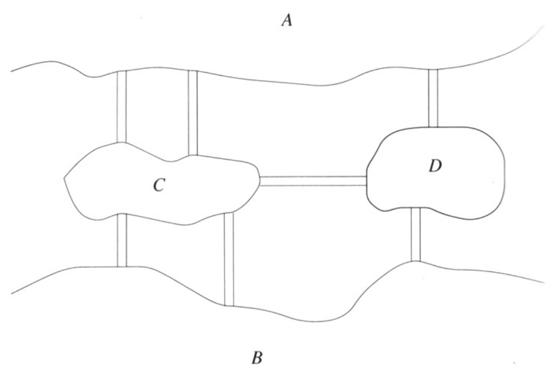


Figure 7.1 The Königsberg bridges problem.

Source: [Manber 1989].

Can one start from one of the lands, cross every bridge exactly once, and return to the origin?

The Königsberg Bridges Problem (cont.)

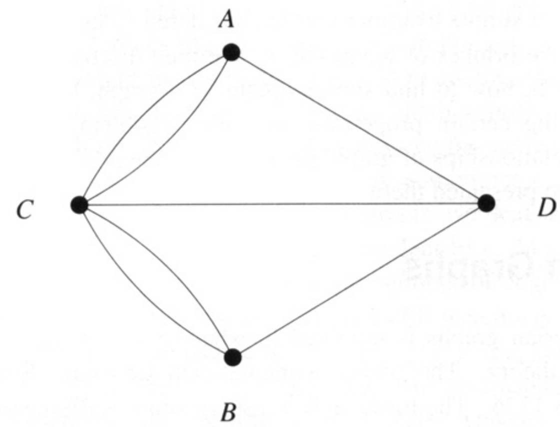


Figure 7.2 The graph corresponding to the Königsberg bridges problem.

Source: [Manber 1989].

Graphs

- A graph consists of a set of vertices (or nodes) and a set of edges (or links, each normally connecting two vertices).
- A graph is commonly denoted as $G(V, E)$, where
  - G is the name of the graph,
  - V is the set of vertices, and
  - E is the set of edges.

Modeling with Graphs

- Reachability
  - Finding program errors
  - Solving sliding tile puzzles
- Shortest Paths
  - Finding the fastest route to a place
  - Routing messages in networks
- Graph Coloring
  - Coloring maps
  - Scheduling classes

Graphs (cont.)

- Undirected vs. Directed Graph
- Simple Graph vs. Multigraph
- Path, Simple Path, Trail
- Circuit, Cycle
- Degree, In-Degree, Out-Degree
- Connected Graph, Connected Components
- Tree, Forest
- Subgraph, Induced Subgraph
- Spanning Tree, Spanning Forest
- Weighted Graph

Eulerian Graphs

Problem 1. *Given an undirected connected graph $G = (V, E)$ such that all the vertices have even degrees, find a circuit P such that each edge of E appears in P exactly once.*

The circuit P in the problem statement is called an *Eulerian circuit*.

Theorem 2. *An undirected connected graph has an Eulerian circuit if and only if all of its vertices have even degrees.*

/* Proof sketch:

(The "only if" part) Suppose the graph has an Eulerian circuit. Each time the circuit enters a vertex, it must also leave the vertex from a different edge. For the first vertex in the circuit, it is left first and entered at last via a different edge. So, every vertex mustt have an even degree.

(The "if" part) The proof is by induction on the number of edges. Note that the graph must contain at least a simple cycle, as every vertex is of an even degree.

Base case: the graph is a simple cycle (with one edge or more). The cycle clearly is an Eulerian circuit.

Inductive step: Remove a simple cycle from the graph. The remaining part of the graph may consist of several separated components. Each component is connected and every vertex in the component also has an even degree. The induction hypothesis applies to each component. Connecting the removed cycle and the Eulerian circuit of each component, we have an Eulerian circuit for the entire graph. */

2 Depth-First Search

Depth-First Search

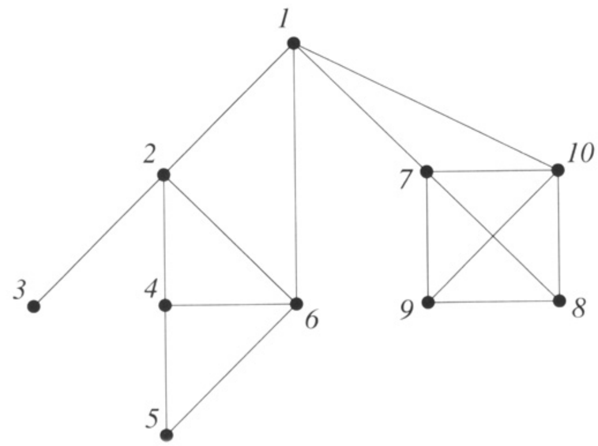


Figure 7.4 A DFS for an undirected graph.

Source: [Manber 1989].

Depth-First Search (cont.)

Algorithm Depth_First_Search(G, v);
begin
  mark v;
  perform preWORK on v;
  for all edges (v, w) **do**
    if w is unmarked **then**
      Depth_First_Search(G, w);
    perform postWORK for (v, w)
end

Depth-First Search (cont.)

Algorithm Refined_DFS(G, v);
begin
  mark v;
  perform preWORK on v;
  for all edges (v, w) **do**
    if w is unmarked **then**
      Refined_DFS(G, w);
    perform postWORK for (v, w);
  perform postWORK_II on v
end

Connected Components

Algorithm Connected_Components(G);
begin
  Component_Number := 1;
  while there is an unmarked vertex v **do**
    Depth_First_Search(G, v)

(preWORK:
$v.Component := Component\_Number$);
$Component\_Number := Component\_Number + 1$

end

Time complexity: $O(|V| + |E|)$.

/* Each edge of the input graph is checked twice (once from each end). The algorithm also has to scan possibly many isolated vertices (in some cases, $|V|$ may be larger than $|E|$). */

DFS Numbers

Algorithm DFS_Numbering(G, v);
begin
$DFS\_Number := 1$;
$Depth\_First\_Search(G, v)$
(preWORK:
$v.DFS := DFS\_Number$;
$DFS\_Number := DFS\_Number + 1$)
end

Time complexity: $O(|E|)$ (assuming the input graph is connected).

The DFS Tree

Algorithm Build_DFS_Tree(G, v);
begin
$Depth\_First\_Search(G, v)$
(postWORK:
if w was unmarked **then**
add the edge (v, w) to T);
end

The DFS Tree (cont.)

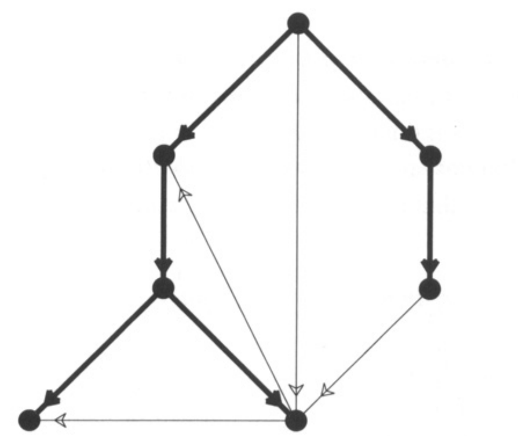


Figure 7.9 A DFS tree for a directed graph.

Source: [Manber 1989].

The DFS Tree (cont.)

Lemma 3 (7.2)**.** *For an undirected graph $G = (V, E)$, every edge $e \in E$ either belongs to the DFS tree T, or connects two vertices of G, one of which is the ancestor of the other in T.*

For undirected graphs, DFS avoids cross edges.

Lemma 4 (7.3)**.** *For a directed graph $G = (V, E)$, if (v, w) is an edge in E such that $v.DFS\_Number < w.DFS\_Number$, then w is a descendant of v in the DFS tree T.*

For directed graphs, cross edges must go "from right to left".

Directed Cycles

Problem 5. *Given a directed graph $G = (V, E)$, determine whether it contains a (directed) cycle.*

Lemma 6 (7.4)**.** *G contains a directed cycle if and only if G contains a back edge (relative to the DFS tree).*

Directed Cycles (cont.)

```
Algorithm Find_a_Cycle(G);
begin
    Depth_First_Search(G, v) /* arbitrary v */
    (preWORK:
        v.on_the_path := true;
     postWORK:
        if w.on_the_path then
            Find_a_Cycle := true;
            halt;
        if w is the last vertex on v's list then
            v.on_the_path := false;)
end
```

Directed Cycles (cont.)

```
Algorithm Refined_Find_a_Cycle(G);
begin
    Refined_DFS(G, v) /* arbitrary v */
    (preWORK:
        v.on_the_path := true;
     postWORK:
        if w.on_the_path then
            Refined_Find_a_Cycle := true;
            halt;
     postWORK_II:
        v.on_the_path := false)
end
```

3 Breadth-First Search

Breadth-First Search

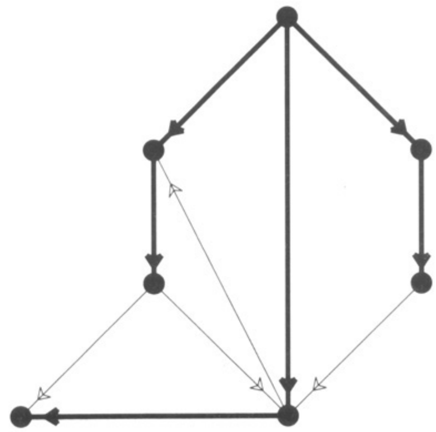


Figure 7.12 A BFS tree for a directed graph.

Source: [Manber 1989].

Breadth-First Search (cont.)

Algorithm Breadth_First_Search(G, v);
begin
  mark v;
  put v in a queue;
  while the queue is not empty **do**
    remove vertex w from the queue;
    perform preWORK on w;
    for all edges (w, x) with x unmarked **do**
      mark x;
      add (w, x) to the BFS tree T;
      put x in the queue
end

Breadth-First Search (cont.)

Lemma 7 (7.5)**.** *If an edge (u, w) belongs to a BFS tree such that u is a parent of w, then u has the minimal BFS number among vertices with edges leading to w.*

Lemma 8 (7.6)**.** *For each vertex w, the path from the root to w in T is a shortest path from the root to w in G.*

Lemma 9 (7.7)**.** *If an edge (v, w) in E does not belong to T and w is on a larger level, then the level numbers of w and v differ by at most 1.*

Breadth-First Search (cont.)

Algorithm Simple_BFS(G, v);
begin
  put v in *Queue*;
  while *Queue* is not empty **do**
    remove vertex w from *Queue*;
    if w is unmarked **then**
      mark w;
      perform preWORK on w;
      for all edges (w, x) with x unmarked **do**
        put x in *Queue*
end

Breadth-First Search (cont.)

Algorithm Simple_Nonrecursive_DFS(G, v);
begin
  push v to *Stack*;
  while *Stack* is not empty **do**
    pop vertex w from *Stack*;
    if w is unmarked **then**
      mark w;
      perform preWORK on w;
      for all edges (w, x) with x unmarked **do**
        push x to *Stack*
end

4 Topological Sorting

Topological Sorting

Problem 10. *Given a directed acyclic graph $G = (V, E)$ with n vertices, label the vertices from 1 to n such that, if v is labeled k, then all vertices that can be reached from v by a directed path are labeled with labels $> k$.*

Lemma 11 (7.8)**.** *A directed acyclic graph always contains a vertex with indegree 0.*

Topological Sorting (cont.)

Algorithm Topological_Sorting(G);
  initialize $v.indegree$ for all vertices; /* by *DFS* */
  $G\_label := 0$;
  for $i := 1$ to n **do**
    if $v_i.indegree = 0$ **then** put v_i in *Queue*;
  repeat
    remove vertex v from *Queue*;
    $G\_label := G\_label + 1$;
    $v.label := G\_label$;
    for all edges (v, w) **do**
      $w.indegree := w.indegree - 1$;
      if $w.indegree = 0$ **then** put w in *Queue*
  until *Queue* is empty

5 Shortest Paths

Single-Source Shortest Paths

Problem 12. *Given a directed graph $G = (V, E)$ and a vertex v, find shortest paths from v to all other vertices of G.*

Shorted Paths: The Acyclic Case

```
Algorithm Acyclic_Shortest_Paths(G, v, n);
{Initially, w.SP = ∞, for every node w.}
{A topological sort has been performed on G, ...}
begin
    let z be the vertex labeled n;
    if z ≠ v then
        Acyclic_Shortest_Paths(G − z, v, n − 1);
        for all w such that (w, z) ∈ E do
            if w.SP + length(w, z) < z.SP then
                z.SP := w.SP + length(w, z)
    else v.SP := 0
end
```

The Acyclic Case (cont.)

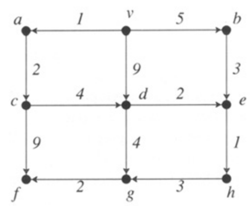
```
Algorithm Imp_Acyclic_Shortest_Paths(G, v);
   for all vertices w do w.SP := ∞;
   initialize v.indegree for all vertices;
   for i := 1 to n do
      if v_i.indegree = 0 then put v_i in Queue;
   v.SP := 0;
   repeat
      remove vertex w from Queue;
      for all edges (w, z) do
         if w.SP + length(w, z) < z.SP then
            z.SP := w.SP + length(w, z);
         z.indegree := z.indegree − 1;
         if z.indegree = 0 then put z in Queue
   until Queue is empty
```

Shortest Paths: The General Case

```
Algorithm Single_Source_Shortest_Paths(G, v);
begin
    for all vertices w do
        w.mark := false;
        w.SP := ∞;
    v.SP := 0;
    while there exists an unmarked vertex do
        let w be an unmarked vertex s.t. w.SP is minimal;
        w.mark := true;
        for all edges (w, z) such that z is unmarked do
```

if $w.SP + length(w, z) < z.SP$ **then**
$z.SP := w.SP + length(w, z)$
end

The General Case (cont.)



	v	a	b	c	d	e	f	g	h
a	0	1	5	∞	9	∞	∞	∞	∞
c	0	1	5	3	9	∞	∞	∞	∞
b	0	1	5	3	7	∞	12	∞	∞
d	0	1	5	3	7	8	12	∞	∞
e	0	1	5	3	7	8	12	11	∞
h	0	1	5	3	7	8	12	11	9
g	0	1	5	3	7	8	12	11	9
f	0	1	5	3	7	8	12	11	9

Figure 7.18 An example of the single-source shortest-paths algorithm.

Source: [Manber 1989].

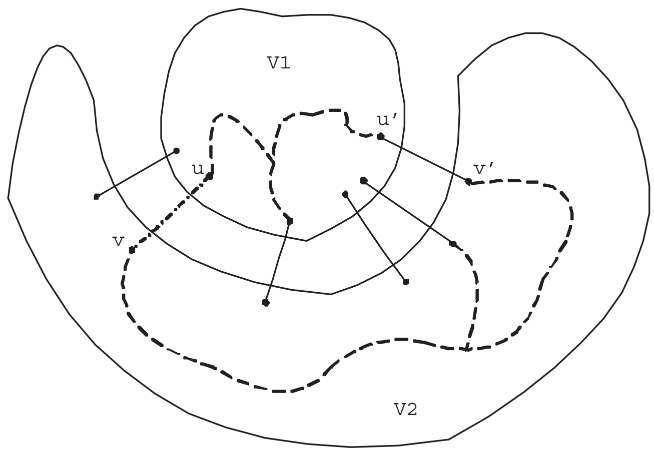
6 Minimum-Weight Spanning Trees

Minimum-Weight Spanning Trees

Problem 13. *Given an undirected connected weighted graph $G = (V, E)$, find a spanning tree T of G of minimum weight.*

Theorem 14. *Let V_1 and V_2 be a partition of V and $E(V_1, V_2)$ be the set of edges connecting nodes in V_1 to nodes in V_2. The edge with the minimum weight in $E(V_1, V_2)$ must be in the minimum-cost spanning tree of G.*

Minimum-Weight Spanning Trees (cont.)



If $cost(u, v)$ is the smallest among $E(V_1, V_2)$, then $\{u, v\}$ must be in the minimum spanning tree.

Minimum-Weight Spanning Trees (cont.)

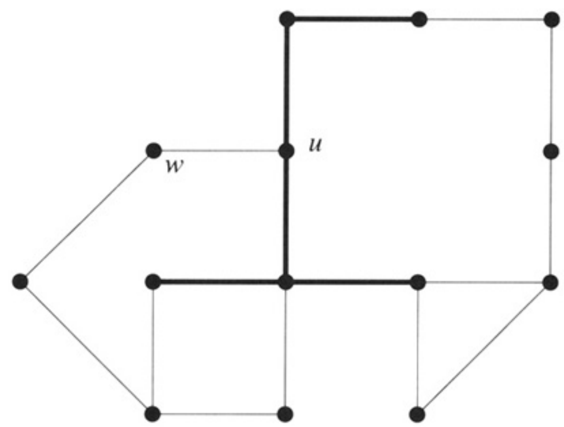


Figure 7.19 Finding the next edge of the MCST.

Source: [Manber 1989].

Minimum-Weight Spanning Trees (cont.)

Algorithm MST(G);
begin
  initially T is the empty set;
  for all vertices w **do**
    $w.mark := false$; $w.cost := \infty$;
  let (x, y) be a minimum cost edge in G;
  $x.mark := true$;
  for all edges (x, z) **do**
    $z.edge := (x, z)$; $z.cost := cost(x, z)$;

Minimum-Weight Spanning Trees (cont.)

```
while there exists an unmarked vertex do
    let w be an unmarked vertex with minimal w.cost;
    if w.cost = ∞ then
        print “G is not connected”;  halt
    else
        w.mark := true;
        add w.edge to T;
        for all edges (w, z) do
            if not z.mark then
                if cost(w, z) < z.cost then
                    z.edge := (w, z);  z.cost := cost(w, z)
end
```

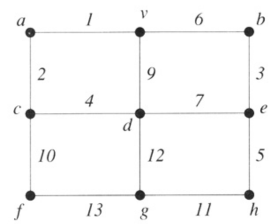
Minimum-Weight Spanning Trees (cont.)

```
Algorithm Another_MST(G);
begin
    initially T is the empty set;
    for all vertices w do
        w.mark := false;  w.cost := ∞;
    x.mark := true; /* x is an arbitrary vertex */
    for all edges (x, z) do
        z.edge := (x, z);  z.cost := cost(x, z);
```

Minimum-Weight Spanning Trees (cont.)

```
    while there exists an unmarked vertex do
        let w be an unmarked vertex with minimal w.cost;
        if w.cost = ∞ then
            print “G is not connected”;  halt
        else
            w.mark := true;
            add w.edge to T;
            for all edges (w, z) do
                if not z.mark then
                    if cost(w, z) < z.cost then
                        z.edge := (w, z);
                        z.cost := cost(w, z)
end
```

Minimum-Weight Spanning Trees (cont.)



	v	a	b	c	d	e	f	g	h
v	-	$v(1)$	$v(6)$	∞	$v(9)$	∞	∞	∞	∞
a	-	-	$v(6)$	$a(2)$	$v(9)$	∞	∞	∞	∞
c	-	-	$v(6)$	-	$c(4)$	∞	$c(10)$	∞	∞
d	-	-	$v(6)$	-	-	$d(7)$	$c(10)$	$d(12)$	∞
b	-	-	-	-	-	$b(3)$	$c(10)$	$d(12)$	∞
e	-	-	-	-	-	-	$c(10)$	$d(12)$	$e(5)$
h	-	-	-	-	-	-	$c(10)$	$h(11)$	-
f	-	-	-	-	-	-	-	$h(11)$	-
g	-	-	-	-	-	-	-	-	-

Figure 7.21 An example of the minimum-cost spanning-tree algorithm.

Source: [Manber 1989].

7 All Shortest Paths

All Shortest Paths

Problem 15. *Given a weighted graph $G = (V, E)$ (directed or undirected) with nonnegative weights, find the minimum-length paths between all pairs of vertices.*

Floyd's Algorithm

Algorithm All_Pairs_Shortest_Paths(W);
begin
  {initialization}
  for $i := 1$ to n **do**
    for $j := 1$ to n **do**
      if $(i, j) \in E$ **then** $W[i, j] := length(i, j)$
      else $W[i, j] := \infty$;
  for $i := 1$ to n **do** $W[i, i] := 0$;

  for $m := 1$ to n **do** {the induction sequence}
    for $x := 1$ to n **do**
      for $y := 1$ to n **do**
        if $W[x, m] + W[m, y] < W[x, y]$ **then**
          $W[x, y] := W[x, m] + W[m, y]$
end

Transitive Closure

Problem 16. *Given a directed graph $G = (V, E)$, find its transitive closure.*

Algorithm Transitive_Closure(A);
begin

```
    {initialization omitted}
    for m := 1 to n do
        for x := 1 to n do
            for y := 1 to n do
                if A[x, m] and A[m, y] then
                    A[x, y] := true
end
```

Transitive Closure (cont.)

```
Algorithm Improved_Transitive_Closure(A);
begin
    {initialization omitted}
    for m := 1 to n do
        for x := 1 to n do
            if A[x, m] then
                for y := 1 to n do
                    if A[m, y] then
                        A[x, y] := true
end
```