Algorithms 2018: Advanced Graph Algorithms

(Based on [Manber 1989])

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1 Strongly Connected Components

Strongly Connected Components

- A directed graph is *strongly connected* if there is a directed path from every vertex to every other vertex.
- A *strongly connected component* (SCC) is a maximal subset of the vertices such that its induced subgraph is strongly connected (namely, there is no other subset that contains it and induces a strongly connected graph).

Strongly Connected Components (cont.)

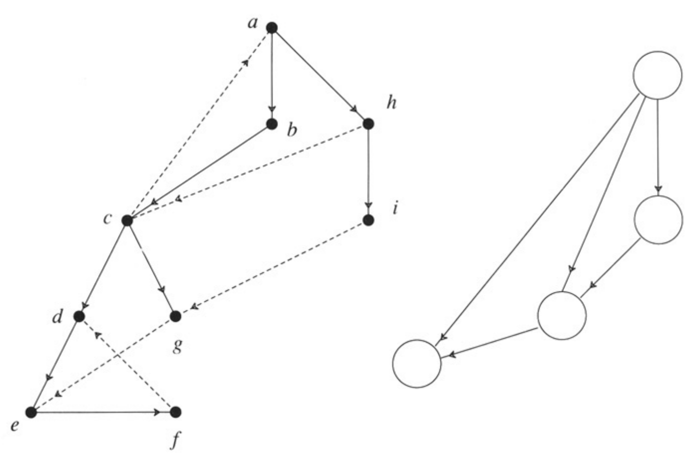


Figure 7.30 A directed graph and its strongly connected component graph.

Source: [Manber 1989].

Strongly Connected Components (cont.)

Lemma 1 (7.11)**.** *Two distinct vertices belong to the same SCC if and only if there is a circuit containing both of them.*

/* An important application of this lemma is that, during a DFS, one vertex will see the other via a back edge (indicating the existence of a directed cycle). This property will be untilized in the algorithm we will study later for determining the SCCs of a graph. */

Lemma 2 (7.12)**.** *Each vertex belongs to exactly one SCC.*

/* All the SCCs of a graph form a partition of the set of vertices of the graph. */

Strongly Connected Components (cont.)

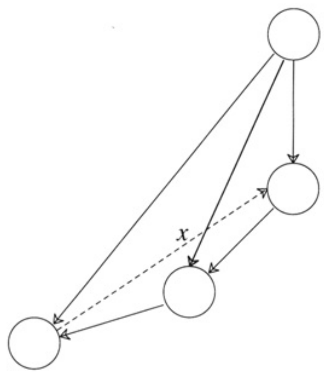


Figure 7.31 Adding an edge connecting two different strongly connected components.

Source: [Manber 1989].

Strongly Connected Components (cont.)

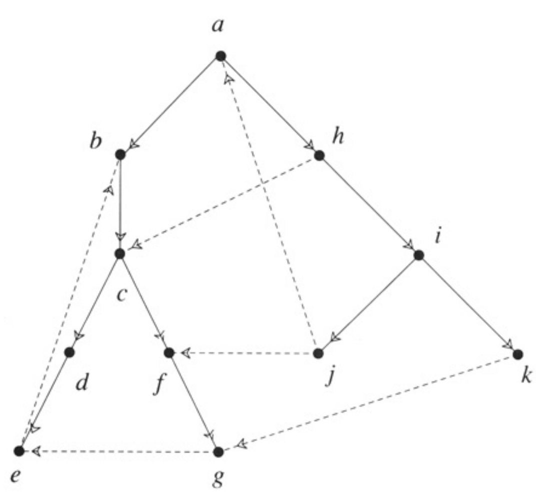


Figure 7.32 The effect of cross edges.

Source: [Manber 1989].

/* A cross edge may point to a vertex in the same SCC under exploration or another SCC that has already been identified. */

Strongly Connected Components (cont.)

Algorithm Strongly_Connected_Components(G, n);
begin
  for every vertex v of G **do**
    $v.DFS\_Number := 0$;
    $v.component := 0$;
  $Current\_Component := 0$; $DFS\_N := n$;
  while $v.DFS\_Number = 0$ for some v **do**
    $SCC(v)$
end

procedure SCC(v);
begin
  $v.DFS\_Number := DFS\_N$;
  $DFS\_N := DFS\_N - 1$;
  insert v into $Stack$;
  $v.high := v.DFS\_Number$;

Strongly Connected Components (cont.)

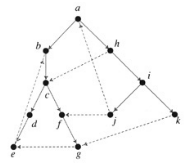
  for all edges (v, w) **do**
    if $w.DFS\_Number = 0$ **then**
      $SCC(w)$;
      $v.high := \max(v.high, w.high)$
    else if $w.DFS\_Number > v.DFS\_Number$
          and $w.component = 0$ **then**
        $v.high := \max(v.high, w.DFS\_Number)$
  if $v.high = v.DFS\_Number$ **then**
    $Current\_Component := Current\_Component + 1$;
    repeat
      remove x from the top of $Stack$;
      $x.component := Current\_Component$
    until $x = v$
end

Time complexity: $O(|E| + |V|)$.

/* This is essentially the DFS with constant extra work per vertex. */

/* For an arbitrary SCC, the vertex v that is visited first during the DFS will acquire the largest/highest DFS number among all the vertices in the same SCC. When the recursive call with v as the input is about to return, v will discover that $v.high = v.DFS\_Number$. */

Strongly Connected Components (cont.)



	a	b	c	d	e	f	g	h	i	j	k
	11	10	9	8	7	6	5	4	3	2	1
a	11	-	-	-	-	-	-	-	-	-	-
b	11	10	-	-	-	-	-	-	-	-	
c	11	10	9	-	-	-	-	-	-	-	-
d	11	10	9	8	-	-	-	-	-	-	-
e	11	10	9	8	10	-	-	-	-	-	-
d	11	10	9	10	10	-	-	-	-	-	-
c	11	10	10	10	10	-	-	-	-	-	-
f	11	10	10	10	10	6	-	-	-	-	-
g	11	10	10	10	10	6	7	-	-	-	-
f	11	10	10	10	10	7	7	-	-	-	-
c	11	10	10	10	10	7	7	-	-	-	-
(b)	11	10	10	10	10	7	7	-	-	-	-
a	11	10	10	10	10	7	7	-	-	-	-
h	11	10	10	10	10	7	7	4	-	-	-
i	11	10	10	10	10	7	7	4	3	-	-
j	11	10	10	10	10	7	7	4	3	11	-
i	11	10	10	10	10	7	7	4	11	11	-
(k)	11	10	10	10	10	7	7	4	11	11	1
i	11	10	10	10	10	7	7	4	11	11	1
h	11	10	10	10	10	7	7	11	11	11	1
(a)	11	10	10	10	10	7	7	11	11	11	1

Figure 7.34 An example of computing *High* values and strongly connected components.

Source: [Manber 1989].

Odd-Length Cycles

Problem 3. *Given a directed graph $G = (V, E)$, determine whether it contains a (directed) cycle of odd length.*

- A cycle must reside completely within a strongly connected component (SCC), so we exam each SCC separately.
- Mark the nodes of an SCC with "even" or "odd" using DFS.
- If we have to mark a node that is already marked in the opposite, then we have found an odd-length cycle.

2 Biconnected Components

Biconnected Components

- An undirected graph is *biconnected* if there are at least two vertex-disjoint paths from every vertex to every other vertex.
- A graph is *not* biconnected if and only if there is a vertex whose removal disconnects the graph. Such a vertex is called an *articulation point*.
- A *biconnected component* (BCC) is a *maximal* subset of the edges such that its induced subgraph is biconnected (namely, there is no other subset that contains it and induces a biconnected graph).

Biconnected Components (cont.)

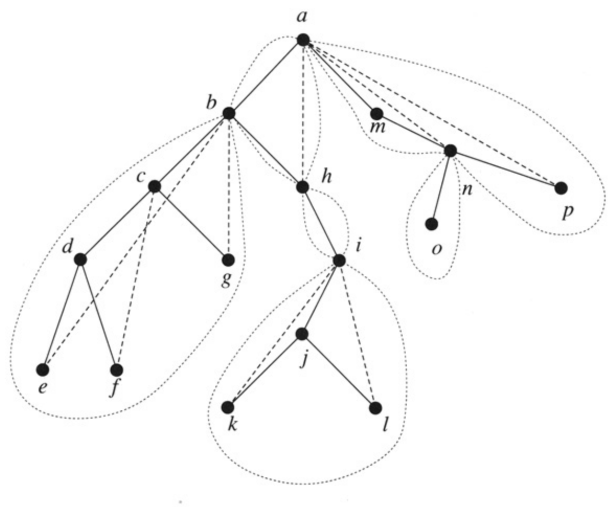


Figure 7.25 The structure of a nonbiconnected graph.

Source: [Manber 1989].

Biconnected Components (cont.)

Lemma 4 (7.9). *Two distinct edges e and f belong to the same BCC if and only if there is a cycle containing both of them.*

Lemma 5 (7.10). *Each edge belongs to exactly one BCC.*

Biconnected Components (cont.)

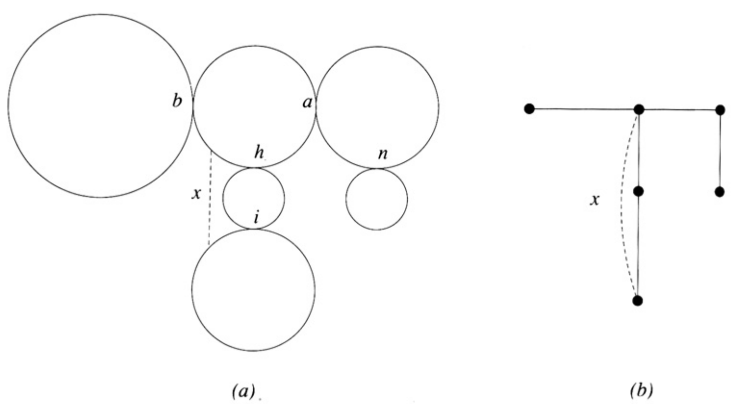


Figure 7.26 An edge that connects two different biconnected components. (a) The components corresponding to the graph of Fig. 7.25 with the articulation points indicated. (b) The biconnected component tree.

Source: [Manber 1989].

Biconnected Components (cont.)

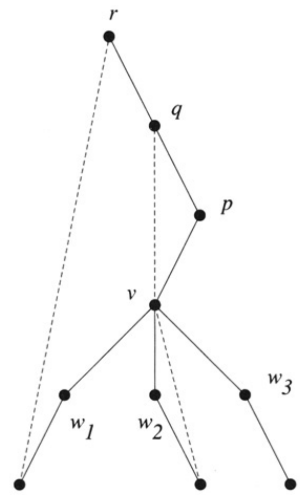


Figure 7.27 Computing the *High* values.

Source: [Manber 1989].

Biconnected Components (cont.)

Algorithm Biconnected_Components(G, v, n);
begin
  for every vertex w **do** $w.DFS\_Number := 0$;
  $DFS\_N := n$;
  $BC(v)$
end

procedure BC(v);
begin
  $v.DFS\_Number := DFS\_N$;
  $DFS\_N := DFS\_N - 1$;
  insert v into $Stack$;
  $v.high := v.DFS\_Number$;

Biconnected Components (cont.)

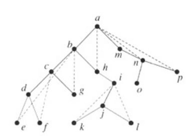
  for all edges (v, w) **do**
    insert (v, w) into $Stack$;
    if w is not the parent of v **then**
      if $w.DFS\_Number = 0$ **then**
        $BC(w)$;
        if $w.high \leq v.DFS\_Number$ **then**
          remove all edges and vertices
            from $Stack$ until v is reached;

insert v back into $Stack$;
$v.high := \max(v.high, w.high)$
else
$v.high := \max(v.high, w.DFS\_Number)$
end

Biconnected Components (cont.)

procedure BC(v);
begin
$v.DFS\_Number := DFS\_N$;
$DFS\_N := DFS\_N - 1$;
$v.high := v.DFS\_Number$;
for all edges (v, w) **do**
if w is not the parent of v **then**
insert (v, w) into $Stack$;
if $w.DFS\_Number = 0$ **then**
$BC(w)$;
if $w.high \leq v.DFS\_Number$ **then**
remove all edges from $Stack$
until (v, w) is reached;
$v.high := \max(v.high, w.high)$
else
$v.high := \max(v.high, w.DFS\_Number)$
end

Biconnected Components (cont.)



	a 16	b 15	c 14	d 13	e 12	f 11	g 10	h 9	i 8	j 7	k 6	l 5	m 4	n 3	o 2	p 1
a	16	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
b	16	15	-	-	-	-	-	-	-	-	-	-	-	-	-	-
c	16	15	14	-	-	-	-	-	-	-	-	-	-	-	-	-
d	16	15	14	13	-	-	-	-	-	-	-	-	-	-	-	-
e	16	15	14	13	15	-	-	-	-	-	-	-	-	-	-	-
d	16	15	14	15	15	-	-	-	-	-	-	-	-	-	-	-
f	16	15	14	15	15	14	-	-	-	-	-	-	-	-	-	-
d	16	15	14	15	15	14	-	-	-	-	-	-	-	-	-	-
c	16	15	15	15	15	14	-	-	-	-	-	-	-	-	-	-
g	16	15	15	15	15	14	15	-	-	-	-	-	-	-	-	-
c	16	15	15	15	15	14	15	-	-	-	-	-	-	-	-	-
(b)	16	15	15	15	14	13	15	-	-	-	-	-	-	-	-	-
h	16	15	15	15	15	14	15	16	-	-	-	-	-	-	-	-
i	16	15	15	15	15	14	15	16	8	-	-	-	-	-	-	-
j	16	15	15	15	15	14	15	16	8	7	-	-	-	-	-	-
k	16	15	15	15	15	14	15	16	8	7	8	-	-	-	-	-
j	16	15	15	15	15	14	15	16	8	8	8	-	-	-	-	-
l	16	15	15	15	15	14	15	16	8	8	8	8	-	-	-	-
j	16	15	15	15	15	14	15	16	8	8	8	8	-	-	-	-
(i)	16	15	15	15	15	14	15	16	8	8	8	8	-	-	-	-
(h)	16	15	15	15	15	14	15	16	8	8	8	8	-	-	-	-
b	16	16	15	15	15	14	15	16	8	8	8	8	-	-	-	-
a	16	16	15	15	15	14	15	16	8	8	8	8	-	-	-	-
m	16	16	15	15	15	14	15	16	8	8	8	8	4	-	-	-
n	16	16	15	15	15	14	15	16	8	8	8	8	4	16	-	-
o	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	-
(n)	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	-
p	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
n	16	16	15	15	15	14	15	16	8	8	8	8	4	16	2	16
m	16	16	15	15	15	14	15	16	8	8	8	8	16	16	2	16
(a)	16	16	15	15	15	14	15	16	8	8	8	8	16	16	2	16

Figure 7.29 An example of computing *High* values and biconnected components.

Source: [Manber 1989].

Even-Length Cycles

Problem 6. *Given a connected undirected graph $G = (V, E)$, determine whether it contains a cycle of even length.*

Theorem 7. *Every biconnected graph that has more than one edge and is not merely an odd-length cycle contains an even-length cycle.*

Even-Length Cycles (cont.)

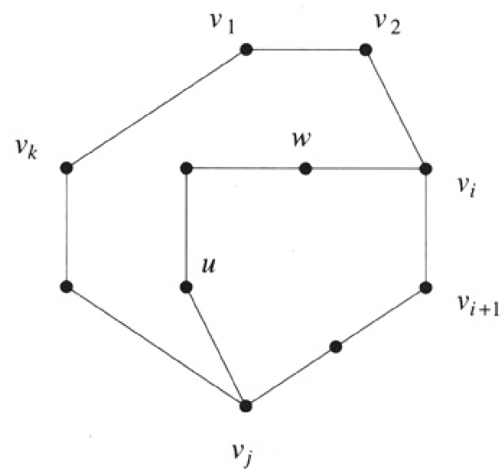


Figure 7.35 Finding an even-length cycle.

Source: [Manber 1989].

3 Network Flows

Network Flows

- Consider a directed graph, or network, $G = (V, E)$ with two distinguished vertices: s (the source) with indegree 0 and t (the sink) with outdegree 0.
- Each edge e in E has an associated positive weight $c(e)$, called the *capacity* of e.

Network Flows (cont.)

- A **flow** is a function f on E that satisfies the following two conditions:
  1. $0 \leq f(e) \leq c(e)$.
  2. $\sum_{u} f(u, v) = \sum_{w} f(v, w)$, for all $v \in V - \{s, t\}$.
- The **network flow problem** is to maximize the flow f for a given network G.

Network Flows (cont.)

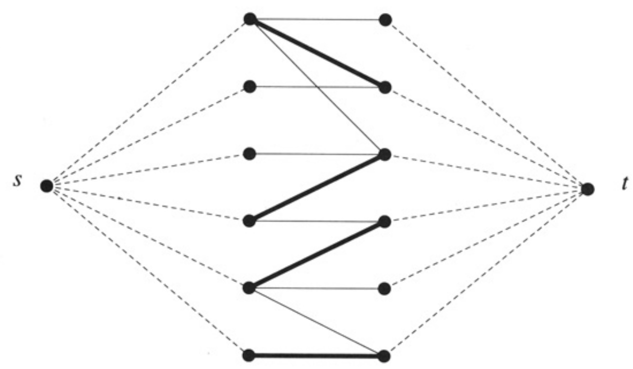


Figure 7.39 Reducing bipartite matching to network flow (the directions of all the edges are from left to right).

Source: [Manber 1989].

Augmenting Paths

- An **augmenting path** w.r.t. a given flow f (of a network G) is a directed path from s to t consisting of edges from G, but not necessarily in the same diretion; each of these edges (v, u) satisfies exactly one of:

  1. (v, u) is in the same direction as it is in G, and $f(v, u) < c(v, u)$. (*forward edge*)
  2. (v, u) is in the opposite direction in G (namely, $(u, v) \in E$), and $f(u, v) > 0$. (*backward edge*)

- If there exists an augmenting path w.r.t. a flow f (f *admits* an augmenting path), then f is not maximum.

Augmenting Paths (cont.)

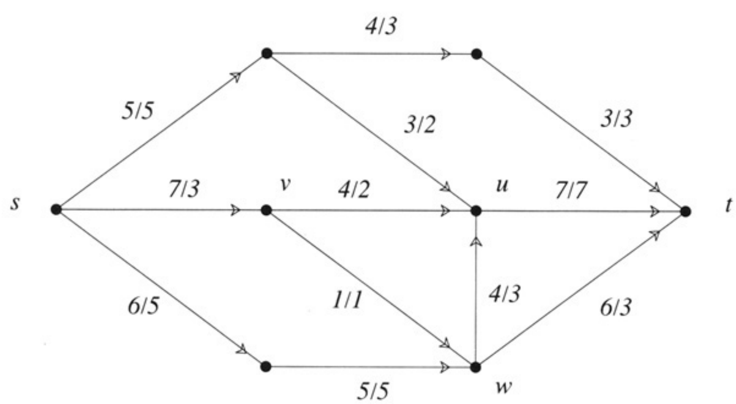


Figure 7.40 An example of a network with a (nonmaximum) flow.

Source: [Manber 1989].

Augmenting Paths (cont.)

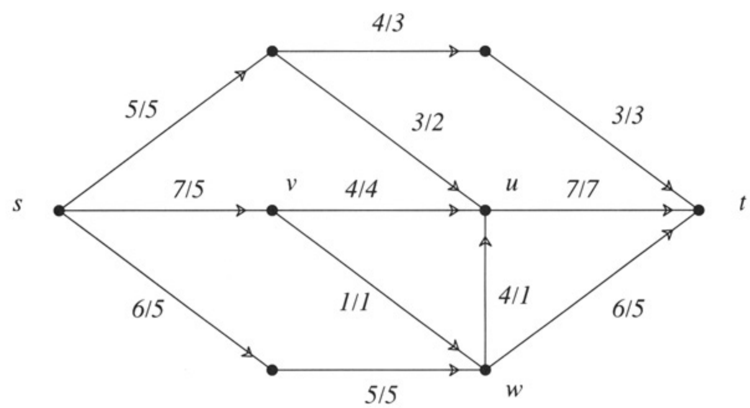


Figure 7.41 The result of augmenting the flow of Fig. 7.40.

Source: [Manber 1989].

Properties of Network Flows

Theorem 8 (Augmenting-Path)**.** *A flow f is maximum if and only if it admits no augmenting path.*

A *cut* is a set of edges that separate s from t, or more precisely a set of the form $\{(v, w) \in E \mid v \in A \text{ and } w \in B\}$, where $B = V - A$ such that $s \in A$ and $t \in B$.

Theorem 9 (Max-Flow Min-Cut)**.** *The value of a maximum flow in a network is equal to the minimum capacity of a cut.*

Properties of Network Flows (cont.)

Theorem 10 (Integral-Flow)**.** *If the capacities of all edges in the network are integers, then there is a maximum flow whose value is an integer.*

Residual Graphs

- The **residual graph** with respect to a network $G = (V, E)$ and a flow f is the network $R = (V, F)$, where F consists of all forward and backward edges and their capacities are given as follows:
    1. $c_R(v, w) = c(v, w) - f(v, w)$ if (v, w) is a forward edge and
    2. $c_R(v, w) = f(w, v)$ if (v, w) is a backward edge.
- An augmenting path is thus a regular directed path from s to t in the residual graph.

Residual Graphs (cont.)

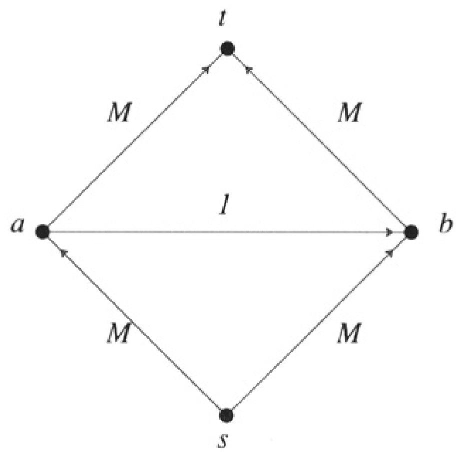


Figure 7.42 A bad example of network flow.

Source: [Manber 1989].