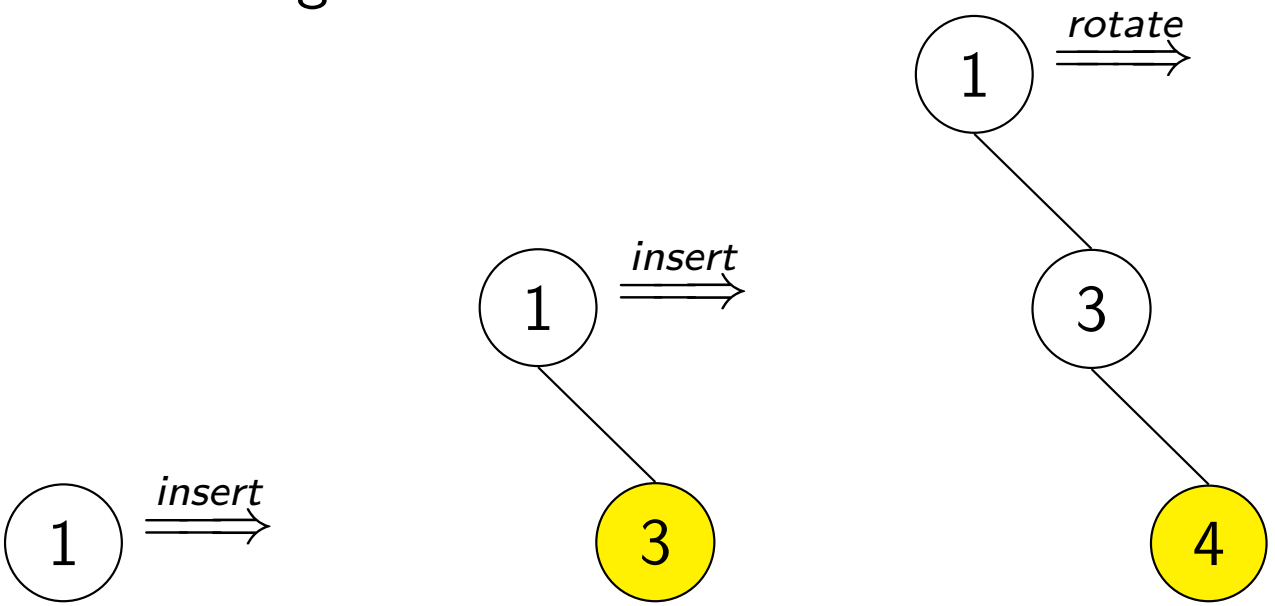
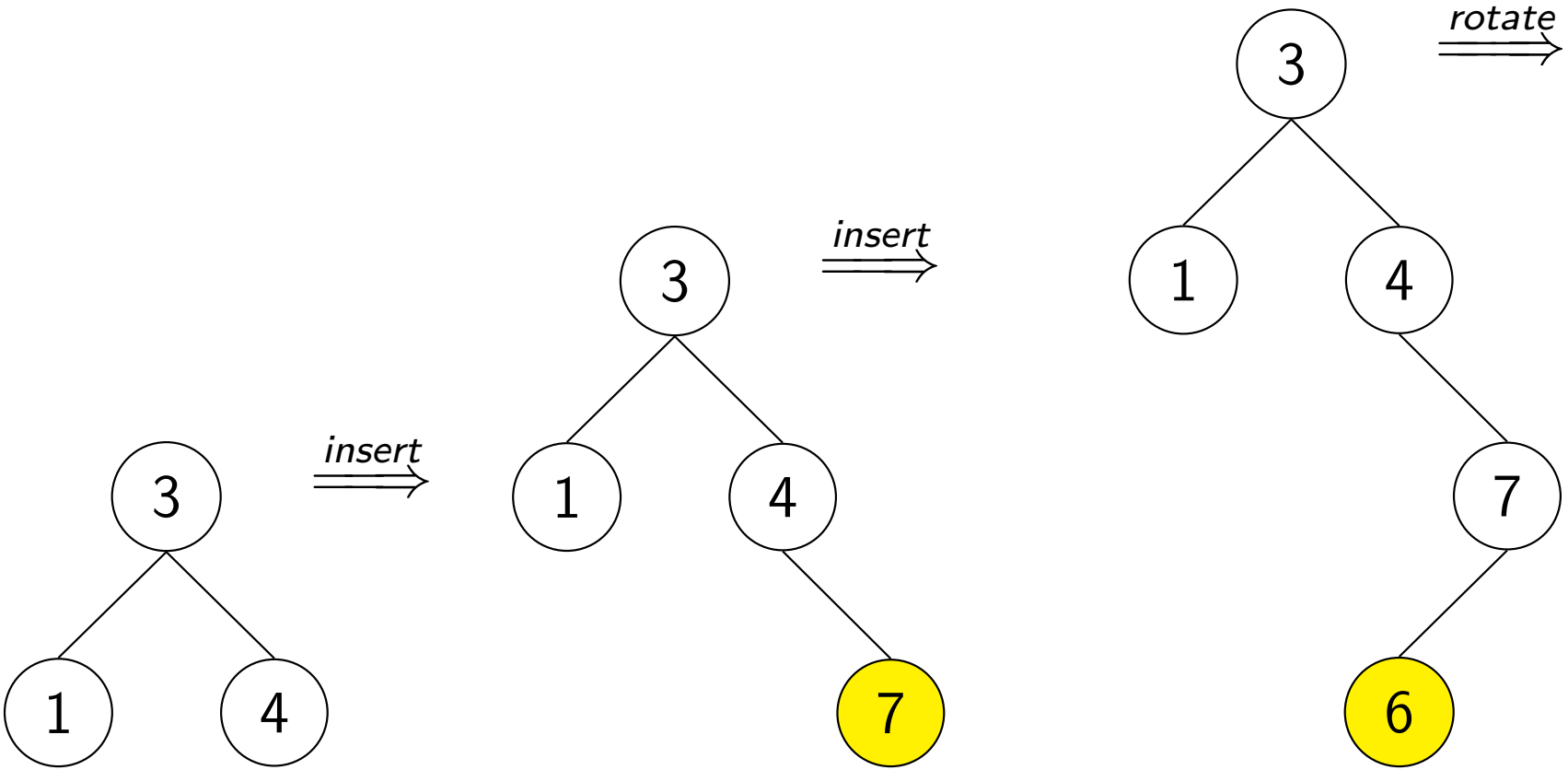
# Homework 6

# Problem1

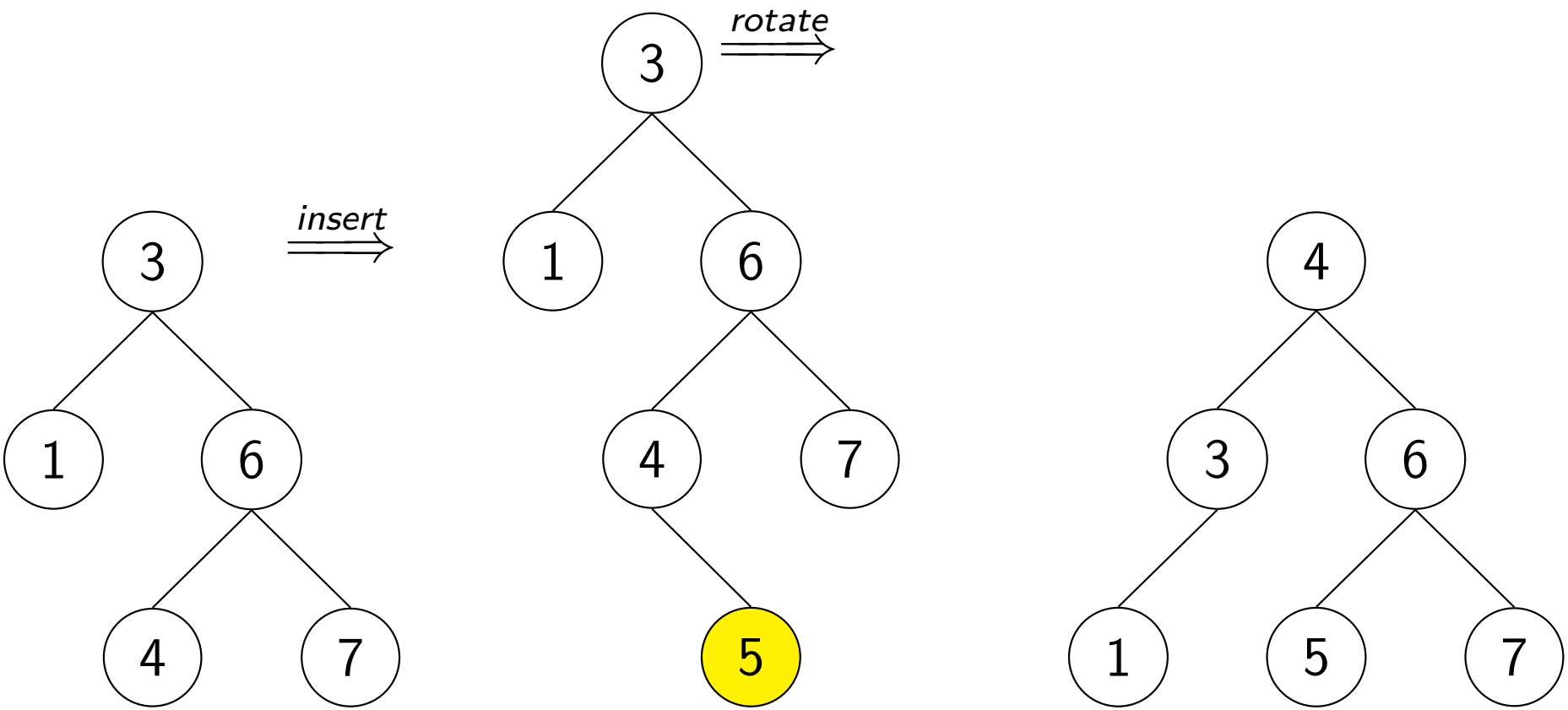
Perform insertions of the numbers 1, 3, 4, 7, 6, 5, 2 (in this order) into an empty AVL tree. Show each AVL tree after a number has been inserted.
If re-balancing operations are performed, please also show the tree before re-balancing and indicate what type of rotation is used in the re-balancing.



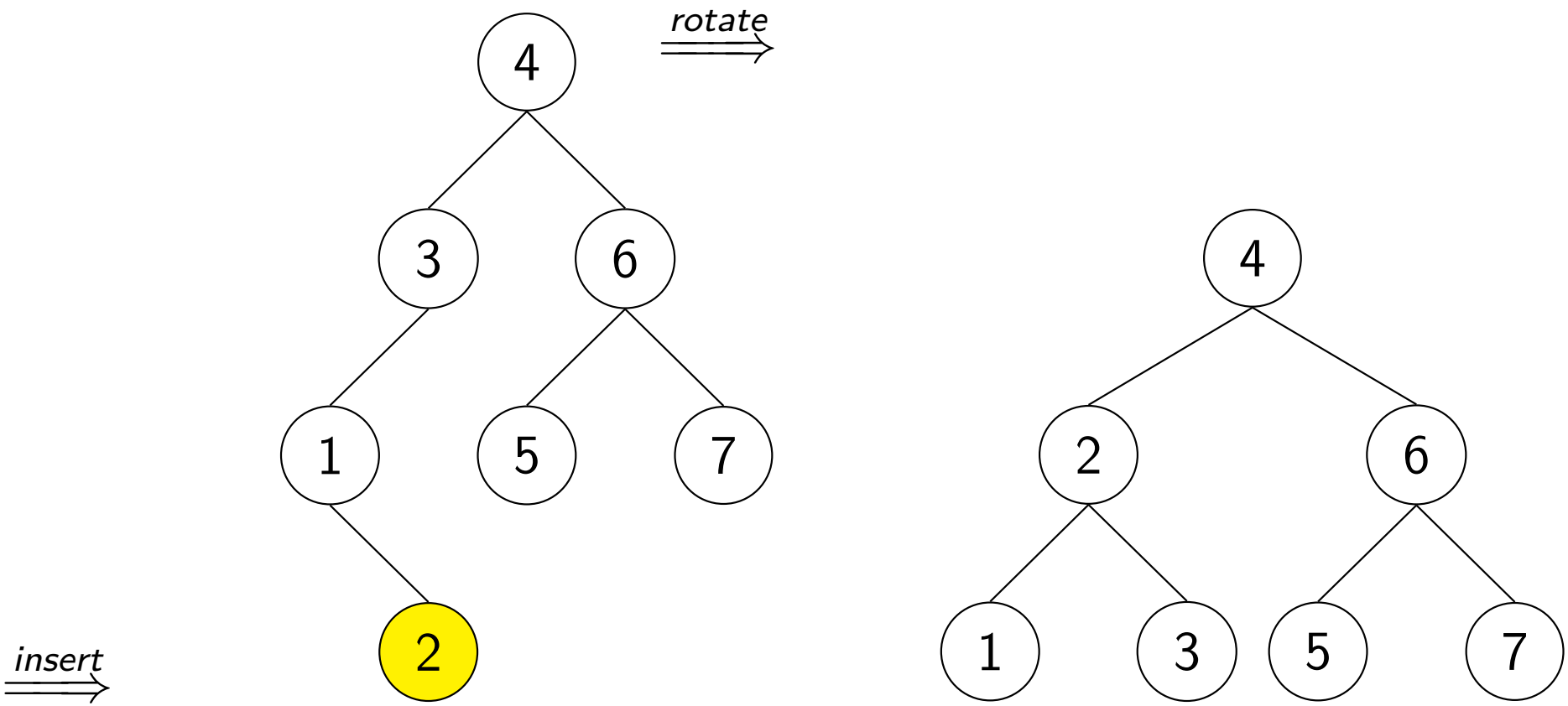
# Problem1



# Problem1



# Problem1



# Problem2

(6.32) Prove that the sum of the heights of all nodes in a complete binary tree with n nodes is at most n - 1.
(A complete binary tree with n nodes is one that can be compactly represented by an array A of size n, where the root is stored in A[1] and the left and the right children of A[i], $1 \leq i \leq \left\lfloor \frac{x}{2} \right\rfloor$ , are stored respectively in A[2i] and A[2i + 1].
Notice that, in Manber′ s book a complete binary tree is referred to as a balanced binary tree and a full binary tree as a complete binary tree. Manber′ s definitions seem to be less frequently used.
Do not let the different names confuse you. "Balanced binary tree" in the original problem description is the same as "complete binary tree" )

# Problem2

- h 表示樹的高度，G(h) 表示在 h 高度下，每個 node 的高度總和
- For a full binary tree (complete binary trees 的一個特例),
  node(n) 數量 $= 2^{h+1} - 1$
  G(h)$= 2^{h+1} - h - 2 = (2^{h+1} - 1) - h - 1$
  $= (2^{h+1} - 1) - (h + 1) = n - (h + 1) \leq n - 1$
  (proven in homework 2)
- With this as a basis, we prove the general case of complete binary trees by induction on the height h ($\geq 0$) of the tree.

# Problem2

- [Base Case](h=0, n=1): G(h)=0 $\leq$ 1-1
- [Inductive step]
  [Induction Hypothesis] The height of sum of complete binary is G(h)$\leq$n-1
  There will be four cases on complete binary tree
  - [Case1]Full binary tree(we have proven this case before)
  - [Case2]Full left subtree and complete right subtree(both height=h)
  - [Case3]Both Left and right subtree are full ($h_L$ = h and $h_R$ = h-1)
  - [Case4]Full right subtree and complete left subtree($h_L$ = h and $h_R$ = h-1)

# Problem2

[Case2]Full left subtree and complete right subtree(both height=h)

$G_c(\text{h}+1)=\text{G}(h_L)+G_c(h_R)+(\text{h}+1)$

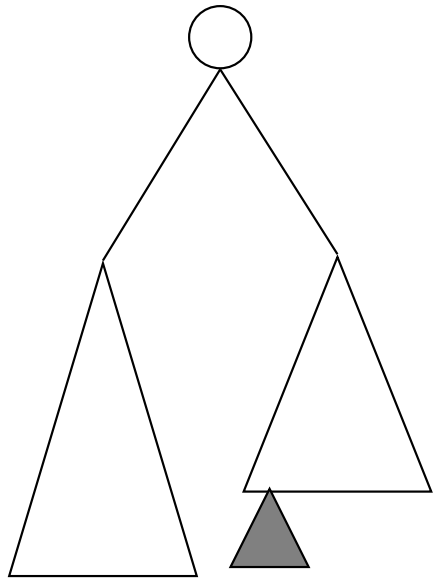
$=[n_L\text{-}(\text{h}+1)]+G_c(h_R)+(\text{h}+1)$

$\leq[n_L\text{-}(\text{h}+1)]+[n_R\text{-}1]+(\text{h}+1)$

$=n_L+n_R\text{-}1$

$=(n_L+n_R+1)\text{-}2$

$=\text{n-}2\leq\text{n-}1$



# Problem2

[Case3]Both Left and right subtree are full ($h_L$ = h and $h_R$ = h-1)

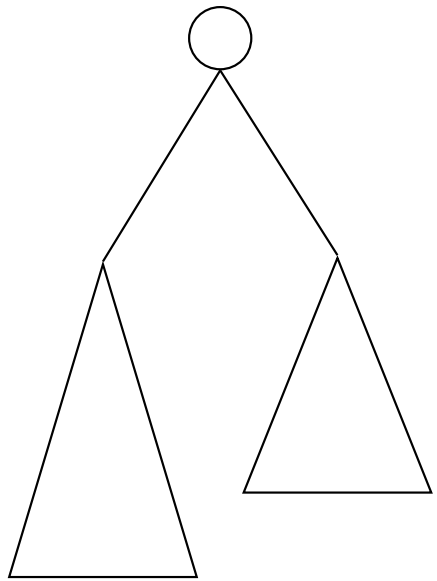
$G_c$(h+1)=G($h_L$)+G($h_R$)+(h+1)

=G(h)+G(h-1)+(h+1)

=[$n_L$-(h+1)]+[$n_R$-(h-1+1)]+(h+1)

=($n_L$+$n_R$+1)-h-1

=n-(h+1)≤n-1



# Problem2

[Case4]Full right subtree and complete left subtree($h_L$ = h and $h_R$ = h-1)

$G_c$(h+1)=$G_c$($h_L$)+G($h_R$)+(h+1)

=$G_c$($h_L$)+G(h-1)+(h+1)

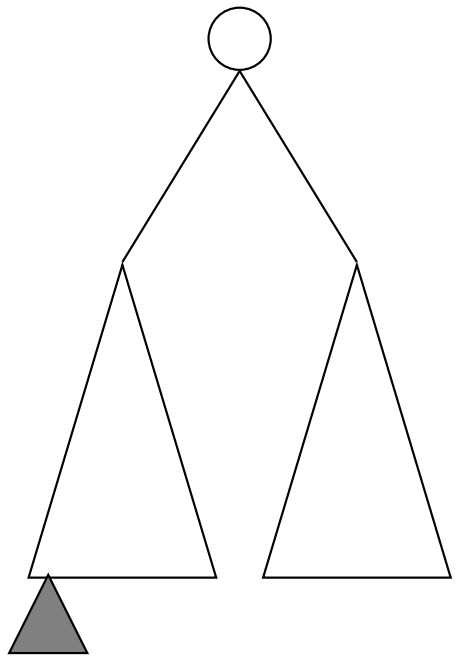
=$G_c$($h_L$)+[$n_R$-(h-1+1)]+(h+1)

≤[$n_L$-1]+[$n_R$-(h-1+1)]+(h+1)

=$n_L$-1+$n_R$+1

=($n_L$+$n_R$+1)-1

=n-1≤n-1



# Problem3

這裡有 12 個硬幣，裡面有最多一個偽幣
偽幣重量和真的不一樣
給你一個天平
用三次秤重就判斷誰是假的，或大家都是真的

# Problem3

善用決策樹
每次秤重，天平會給我們哪幾種資訊？
一樣重、不一樣重：兩種？
$2^3 = 8 < 12 + 1$
一樣重、左邊重、右邊重：三種
$3^3 = 27 > 12 + 1$

# Problem3

從頭想解法不容易
善用決策樹的節點數能少走彎路

一開始想要左六右六秤？
一樣重：完美；某邊重？
因為不知道偽幣是比較重還是比較輕，沒有硬幣能擺脫嫌疑
此時仍有 12 種可能，但你只剩兩次比較，$3^2 = 9 < 12$
左五右五也類似

# Problem3

參考解法
首先左四右四秤，比如，$c_1c_2c_3c_4/c_5c_6c_7c_8$

如果一樣重，這 8 個硬幣擺脫嫌疑
剩下 4 個硬幣，拿 $c_9c_{10}c_{11}$ 出來與任三個真幣秤，比如 $c_9c_{10}c_{11}/c_1c_2c_3$
如果這兩邊一樣重，剩下 $c_{12}$ 有嫌疑，比一下就知道了
如果 $c_9c_{10}c_{11}$ 比較輕/重，不但知道偽幣存在，還知道偽幣是比較輕/重
比 $c_9/c_{10}$，比較輕/重一方就是假的；若兩邊一樣就是 $c_{11}$

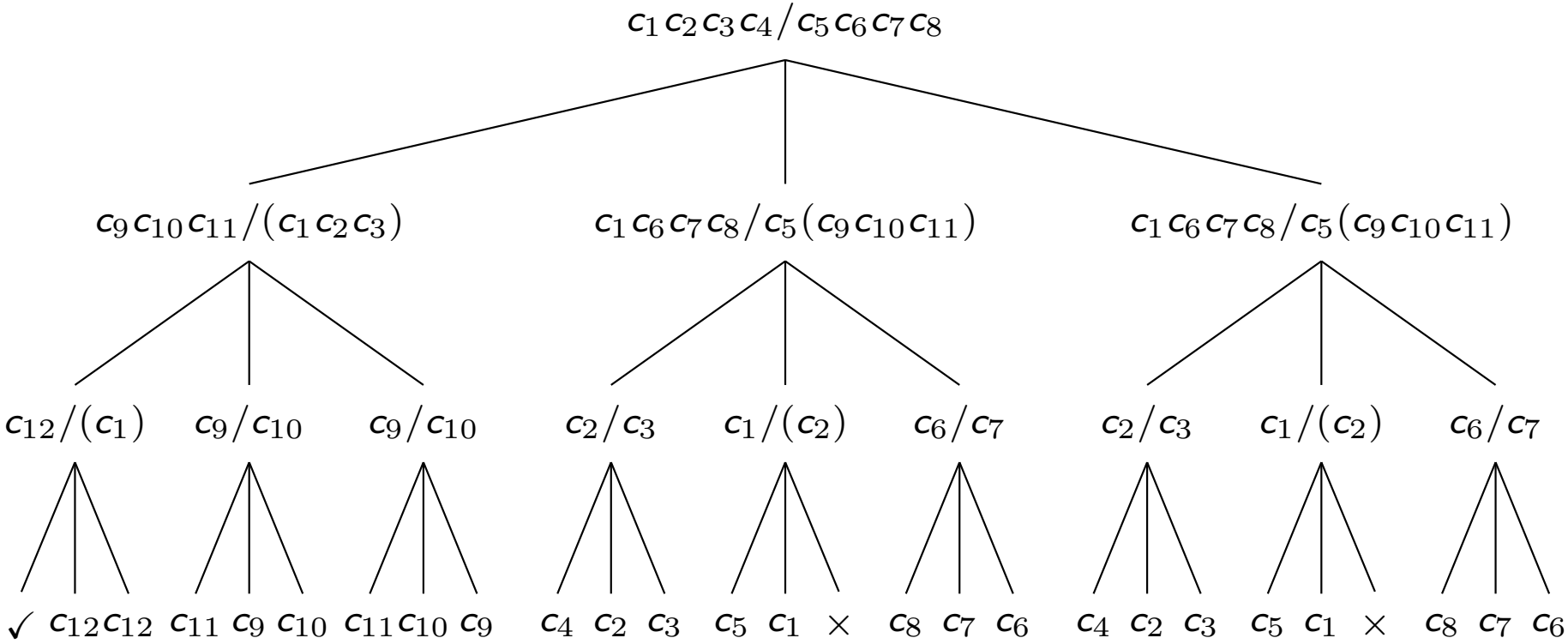
# Problem3

如果 $c_1c_2c_3c_4/c_5c_6c_7c_8$ 不一樣重？
此時我們確定裡頭有偽幣，嫌疑人 8 位
我們要在剩下兩次比較解析出兇手
第二次秤重必須將 8 個嫌疑人分成三部分，$3/3/2$，才有機會在最後一次秤重找到兇手

將 $c_6c_7c_8$ 移到左方，右邊補上三個真幣 $c_9c_{10}c_{11}$
$c_1c_6c_7c_8/c_5c_9c_{10}c_{11}$
如果傾斜的方向一樣，代表問題出在 $c_1c_5$，拿 $c_1$ 與真幣秤就行
如果兩邊一樣重，代表問題在 $c_2c_3c_4$；如果傾斜方向反了，問題在 $c_6c_7c_8$
而且能夠從第一次秤重的結果判斷偽幣輕/重
剩下的三個嫌疑人其中兩個互相秤，輕/重的一方為假，否則第三個是假

# Problem3

為了更清楚展示這個流程，畫出天平秤重的決策樹：
三個子節點分別代表：一樣重/左邊重/右邊重
括號代表當下已經確認為真貨的硬幣



# Problem3

為什麼不能用 2 次比較？
因為 $3^2 = 9 < 12 + 1$

可能性只有 13 種嗎？
更好的答案是：$3^2 = 9 < 24 + 1$

由於這個問題只要我們判斷「誰是假的」，所以只需要 13 個節點
但解題過程中用到的天平使得我們依賴於偽幣的輕重，而一開始這個訊息是未知的
我們需要兩倍：25 個節點來完整表達：誰是偽幣、比較輕還是比較重
無論是 13 還是 25，都無法用兩次比較達成

# Problem4

Design an algorithm that, given a set of integers $S = \{x_1, x_2, \cdots, x_n\}$, finds a nonempty subset $R \subseteq S$, such that

$$\sum_{x_i \in R} x_i \equiv 0 \pmod{n}.$$

Before presenting your algorithm, please argue why such a nonempty subset must exist.

# Problem4(cont'd)

Define

$$
\begin{aligned}
Sum_1 &= x_1 \\
Sum_2 &= x_1 + x_2 \\
Sum_3 &= x_1 + x_2 + x_3 \\
&\cdots \\
Sum_n &= x_1 + x_2 + x_3 + \cdots + x_n
\end{aligned}
$$

If there exists an $Sum_i \equiv 0 \pmod{n}$ then done. If no, there must be at least two of them have the same remainder because there are $n$ items, $n-1$ kinds of remainders(from $1$ to $n-1$), by pigeonhole principle. Suppose $Sum_i$ and $Sum_j$ $(i < j)$ have the same remainder, then $\{x_{i+1}, x_{i+2}, \cdots, x_j\}$ is what we want since we remove the elements from $\{x_1, x_2, \cdots, x_j\}$ ,which cause the remainder. $\{x_{i+1}, x_{i+2}, \cdots, x_j\}$ cannot be empty because $i \neq j$.

# Problem4(cont'd)

**function** FIND SUBSET($S$, $n$)
    $R_1 := x_1 \mod n$ ▷ $R$ represents the remainder of $Sum$
    **for** $i$ from $1$ to $n-1$ **do**
        $R_i := (R_{i-1} + x_i) \mod n$ ▷ Compute $Sum_i \mod n$
        **if** $R_i == 0$ **then**
            print $\{x_1, x_2, \cdots, x_i\}$
            return
    i, j := FindIdentical(R) ▷ Get index of two items with the same remainder
    print $\{x_{i+1}, x_{i+2}, \cdots, x_j\}$

# Problem5

Consider the *next* table as in the KMP algorithm for string $B[1..9] =$ *abaababaa*.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| a | b | a | a | b | a | b | a | a |
| -1 | 0 | 0 | 1 | 1 | 2 | 3 | 2 | 3 |

Suppose that, during an execution of the KMP algorithm, $B[6]$ (which is an *a*) is being compared with a letter in $A$, say $A[i]$, which is not an *a* and so the matching fails. The algorithm will next try to compare $B[next[6] + 1]$, i.e., $B[3]$ which is also an *a*, with $A[i]$. The matching is bound to fail for the same reason. This comparison could have been avoided, as we know from $B$ itself that $B[6]$ equals $B[3]$ and, if $B[6]$ does not match $A[i]$, then $B[3]$ certainly will not, either. $B[5]$, $B[8]$, and $B[9]$ all have the same problem, but $B[7]$ does not. Please adapt the computation of the next table so that such wasted comparisons can be avoided.

# Problem5(cont'd)

**function** NEW NEXT($B$, $m$, $next$)
    **for** $i$ from $3$ to m **do**
        **if** $B[next[i]+1] == B[i]$ **then**
            $next[i] := next[next[i]+1]$

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| a | b | a | a | b | a | b | a | a |
| -1 | 0 | -1 | 1 | 0 | -1 | 3 | -1 | 1 |