Analysis of Algorithms
(Based on [Manber 1989])

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Introduction

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- The purpose of algorithm analysis is to predict the behavior (running time, space requirement, etc.) of an algorithm *without implementing it* on a specific computer. (Why?)
- As the exact behavior of an algorithm is hard to predict, the analysis is usually an *approximation*:
  - Relative to the input size (usually denoted by n): input possibilities too enormous to elaborate
  - Asymptotic: should care more about larger inputs
  - Worst-Case: easier to do, often representative (Why not average-case?)
- Such an approximation is usually good enough for comparing different algorithms for the same problem.

Complexity

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- Theoretically, “complexity of an algorithm” is a more precise term for “approximate behavior of an algorithm”.
- Two most important measures of complexity:
  - Time Complexity
    an upper bound on the number of steps that the algorithm performs.
  - Space Complexity
    an upper bound on the amount of temporary storage required for running the algorithm (excluding the input, the output, and the program itself).
- We will focus on time complexity.

Comparing Running Times

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  1. $100n$
  2. $2n^2 + 50$
  3. $100n^{1.8}$

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- We will study an approach (the O notation) that allows us to ignore constant factors and concentrate on the behavior as n goes to infinity.
- For most algorithms, the constants in the expressions of their running times tend to be small.

The O Notation

- A function $g(n)$ is $O(f(n))$ for another function $f(n)$ if there exist constants c and N such that, for all $n \geq N$, $g(n) \leq cf(n)$.
- The function $g(n)$ may be substantially less than $cf(n)$; the O notation bounds it *only from above*.
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- The function $g(n)$ may be substantially less than $cf(n)$; the O notation bounds it *only from above*.
- The O notation allows us to ignore constants conveniently.
- Examples:
  - $5n^2 + 15 = O(n^2)$.
    (cf. $5n^2 + 15 \leq O(n^2)$ or $5n^2 + 15 \in O(n^2)$)
  - $5n^2 + 15 = O(n^3)$.
    (cf. $5n^2 + 15 \leq O(n^3)$ or $5n^2 + 15 \in O(n^3)$)
  - As part of an expression like $T(n) = 3n^2 + O(n)$.

The O Notation (cont.)

- No need to specify the base of a logarithm.
  - $\log_2 n = \frac{\log_{10} n}{\log_{10} 2} = \frac{1}{\log_{10} 2} \log_{10} n$.
  - For example, we can just write $O(\log n)$.
- $O(1)$ denotes a constant.

Properties of O

- We can add and multiply with O.

Lemma (3.2)

1. If $f(n) = O(s(n))$ and $g(n) = O(r(n))$, then $f(n) + g(n) = O(s(n) + r(n))$.
2. If $f(n) = O(s(n))$ and $g(n) = O(r(n))$, then $f(n) \cdot g(n) = O(s(n) \cdot r(n))$.

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- However, we cannot subtract or divide with O. (Why?)
  - $2n = O(n)$, $n = O(n)$, and $2n - n = n \neq O(n - n)$.
  - $n^2 = O(n^2)$, $n = O(n^2)$, and $n^2/n = n \neq O(n^2/n^2)$.

Polynomial vs. Exponential

- A function $f(n)$ is *monotonically growing* (or *monotonically increasing*) if $n_1 \geq n_2$ implies that $f(n_1) \geq f(n_2)$.
- An exponential function grows *at least* as fast as a polynomial function does.

Theorem (3.1)

For all constants $c > 0$ and $a > 1$, and for all monotonically growing functions $f(n)$, $(f(n))^c = O(a^{f(n)})$.

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- Examples:
  - Take n as $f(n)$, $n^c = O(a^n)$.
  - Take $\log_a n$ as $f(n)$, $(\log_a n)^c = O(a^{\log_a n}) = O(n)$.

Speed of Growth

$\log n$	n	$n \log n$	n^2	n^3	2^n
0	1	0	1	1	2
1	2	2	4	8	4
2	4	8	16	64	16
3	8	24	64	512	256
4	16	64	256	4,096	65,536
5	32	160	1,024	32,768	4,294,967,296

Table: Function values.

Source: redrawn from [E. Horowitz *et al.* 1998, Table 1.7].

Speed of Growth (cont.)

running times	$time_1$ 1000 steps/sec	$time_2$ 2000 steps/sec	$time_3$ 4000 steps/sec	$time_4$ 8000 steps/sec
$\log n$	0.010	0.005	0.003	0.001
n	1	0.5	0.25	0.125
$n \log n$	10	5	2.5	1.25
$n^{1.5}$	32	16	8	4
n^2	1000	500	250	125
n^3	1,000,000	500,000	250,000	125,000
1.1^n	10^{39}	10^{39}	10^{38}	10^{38}

Table: Running times (in seconds) under different assumptions (n = 1000).

Source: redrawn from [Manber 1989, Table 3.1].

O, o, Ω, and Θ

- Let $T(n)$ be the number of steps required to solve a given problem for input size n.
- We say that $T(n) = \Omega(g(n))$ or the problem has a lower bound of $\Omega(g(n))$ if there exist constants c and N such that, for all $n \geq N$, $T(n) \geq cg(n)$.
- If a certain function $f(n)$ satisfies both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, then we say that $f(n) = \Theta(g(n))$.

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- We say that $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

Polynomial vs. Exponential (cont.)

- An exponential function grows *faster* than a polynomial function does.

Theorem (3.3)

For all constants $c > 0$ and $a > 1$, and for all monotonically growing functions $f(n)$, we have

$$(f(n))^c = o(a^{f(n)}).$$

- Consider a previous example again:
  Take $\log_a n$ as $f(n)$. For all $c > 0$ and $a > 1$,

$$(\log_a n)^c = o(a^{\log_a n}) = o(n).$$

Sums

- Techniques for summing expressions are essential for complexity analysis.
- For example, given that we know

$$S_0(n) = \sum_{i=1}^{n} 1 = n$$

and

$$S_1(n) = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

we want to compute the sum

$$S_2(n) = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

Sums (cont.)

From

$$(i+1)^3 = i^3 + 3i^2 + 3i + 1,$$

we have

$$(i+1)^3 - i^3 = 3i^2 + 3i + 1.$$

$$\begin{array}{rcl}
2^3 - 1^3 & = & 3 \times 1^2 + 3 \times 1 + 1 \\
3^3 - 2^3 & = & 3 \times 2^2 + 3 \times 2 + 1 \\
4^3 - 3^3 & = & 3 \times 3^2 + 3 \times 3 + 1 \\
\cdots & \cdots & \cdots \\
(n+1)^3 - n^3 & = & 3 \times n^2 + 3 \times n + 1 \\
\hline
(n+1)^3 - 1 & = & 3 \times S_2(n) + 3 \times S_1(n) + S_0(n) \\
(S_3(n+1) - S_3(1)) - S_3(n) & = & 3 \times S_2(n) + 3 \times S_1(n) + S_0(n)
\end{array}$$

Sums (cont.)

- So, we have

$$(n+1)^3 - 1 = 3 \times S_2(n) + 3 \times S_1(n) + S_0(n).$$

- Given $S_0(n)$ and $S_1(n)$, the sum $S_2(n)$ can be computed by straightforward algebra.
- Recall that the left-hand side $(n+1)^3 - 1$ equals $(S_3(n+1) - S_3(1)) - S_3(n)$, a result from "shifting and canceling" terms of two sums.

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- This generalizes to $S_k(n)$, for $k > 2$.
- Similar shifting and canceling techniques apply to other kinds of sums.

Recurrence Relations

- A *recurrence relation* is a way to define a function by an expression involving the same function.
- The Fibonacci numbers, for example, can be defined as follows:

$$\begin{cases} F(1) = 1 \\ F(2) = 1 \\ F(n) = F(n-2) + F(n-1) \end{cases}$$

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  We would need $k-2$ steps to compute $F(k)$.
- It is more convenient to have an explicit (or closed-form) expression.
- To obtain the explicit expression is called *solving* the recurrence relation.

Guessing and Proving an Upper Bound

- Recurrence relation: $\begin{cases} T(2) = 1 \\ T(2n) \leq 2T(n) + 2n - 1 \end{cases}$
- Guess: $T(n) = O(n \log n)$.

Guessing and Proving an Upper Bound

- Recurrence relation: $\begin{cases} T(2) = 1 \\ T(2n) \leq 2T(n) + 2n - 1 \end{cases}$
- Guess: $T(n) = O(n \log n)$.
- Proof:
  1. Base case: $T(2) \leq 2 \log 2$.
  2. Inductive step: $\begin{aligned}[t] T(2n) &\leq 2T(n) + 2n - 1 \\ &\leq 2(n \log n) + 2n - 1 \\ &= 2n \log n + 2n \log 2 - 1 \\ &\leq 2n(\log n + \log 2) \\ &= 2n \log 2n \end{aligned}$

Solving the Fibonacci Relation

- We will study two techniques for solving the Fibonacci relation.
  1. One uses the characteristic equation
  2. The other uses generating functions
- These techniques may be generalized to handle recurrence relations of the form

$$F(n) = b_1 F(n-1) + b_2 F(n-2) + \cdots + b_k F(n-k)$$

for a constant k.

Using the Characteristic Equation

- $F(n)$ nearly doubles every time and should be an exponential function.
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Using the Characteristic Equation

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- But what is the base of the exponential function?
- The base a should satisfy $a^n = a^{n-1} + a^{n-2}$, which implies $a^2 = a + 1$ (called the characteristic equation).
- There are two solutions to the characteristic equation: $a_1 = \frac{1+\sqrt{5}}{2}$ and $a_2 = \frac{1-\sqrt{5}}{2}$.
- Any linear combination of a_1^n and a_2^n solves the recurrence relation.

Using the Characteristic Equation (cont.)

- So, the general solution is

$$c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Using the Characteristic Equation (cont.)

- So, the general solution is

$$c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

- To fit the values of $F(1)$ and $F(2)$, c_1 and c_2 must satisfy

$$\begin{array}{c} c_1(\frac{1+\sqrt{5}}{2}) + c_2(\frac{1-\sqrt{5}}{2}) = 1 \\ c_1(\frac{1+\sqrt{5}}{2})^2 + c_2(\frac{1-\sqrt{5}}{2})^2 = 1 \end{array}$$

- Therefore, $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$, and the exact solution to the Fibonacci relation is

$$F(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Using Generating Functions

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- *Generating functions* provide a systematic, effective means for representing and manipulating infinite sequences (of numbers).
- We use them here to derive a closed-form representation of the Fibonacci numbers.
- Below are two basic generating functions:

gen. func.	power series	generated sequence
$\frac{1}{1-z}$	$1 + z + z^2 + \cdots + z^n + \cdots$	$1, 1, 1, \cdots, 1, \cdots$
$\frac{c}{1-az}$	$c + caz + ca^2z^2 + \cdots + ca^nz^n + \cdots$	$c, ca, ca^2, \cdots, ca^n, \cdots$

- The second one is a generalization of the first and will be used in our solution.

Using Generating Functions (cont.)

Let $F(z) = 0 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots + F_n z^n + \cdots$ (a generating function for the Fibonacci numbers; $F(n)$ is written as F_n here).

$$
\begin{array}{rcl}
F(z) & = & F_1 z + F_2 z^2 + F_3 z^3 + \cdots + F_n z^n + F_{n+1} z^{n+1} + \cdots \\
zF(z) & = & F_1 z^2 + F_2 z^3 + \cdots + F_{n-1} z^n + F_n z^{n+1} + \cdots \\
z^2 F(z) & = & F_1 z^3 + F_2 z^4 + \cdots + F_{n-2} z^n + F_{n-1} z^{n+1} + \cdots \\
\hline
(1 - z - z^2) F(z) & = & z
\end{array}
$$

$$
\begin{array}{rcl}
F(z) & = & \frac{z}{1-z-z^2} \quad \left(= \frac{z}{(1-\frac{1+\sqrt{5}}{2}z)(1-\frac{1-\sqrt{5}}{2}z)}\right) \\
& = & \frac{\frac{1}{\sqrt{5}}}{1-\frac{1+\sqrt{5}}{2}z} + \frac{-\frac{1}{\sqrt{5}}}{1-\frac{1-\sqrt{5}}{2}z}
\end{array}
$$

Therefore, $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$.

Divide and Conquer Relations

- The running time $T(n)$ of a divide-and-conquer algorithm satisfies
$$T(n) = aT(n/b) + O(n^k)$$
where
  - a is the number of subproblems,
  - n/b is the size of each subproblem, and
  - $O(n^k)$ is the time spent on dividing the problem and combining the solutions.

Divide and Conquer Relations (cont.)

Assume, for simplicity, $n = b^m$ ($\frac{n}{b^m} = 1$, $\frac{n}{b^{m-1}} = b$, etc.).

$$\begin{aligned}
T(n) &= aT(\tfrac{n}{b}) + O(n^k) \\
&= a(aT(\tfrac{n}{b^2}) + O((\tfrac{n}{b})^k)) + O(n^k) \\
&= a(a(aT(\tfrac{n}{b^3}) + O((\tfrac{n}{b^2})^k)) + O((\tfrac{n}{b})^k)) + O(n^k) \\
&\cdots \\
&= a(a(\cdots(aT(\tfrac{n}{b^m}) + O((\tfrac{n}{b^{m-1}})^k)) + \cdots) + O((\tfrac{n}{b})^k)) + O(n^k)
\end{aligned}$$

Assuming $T(1) = O(1)$ (and recalling $n = b^m$, i.e., $m = \log_b n$),

$$T(n) = a^m \times O(1) + \sum_{i=1}^{m} a^{m-i} O(b^{ik}) = O(a^m) + a^m \sum_{i=1}^{m} O((\frac{b^k}{a})^i).$$

Divide and Conquer Relations (cont.)

As $m = \log_b n$ and $a^m = a^{\log_b n} = n^{\log_b a}$,

$$T(n) = O(n^{\log_b a}) + O(n^{\log_b a}) \times O(\sum_{i=1}^{\log_b n} (\frac{b^k}{a})^i).$$

- $O(n^{\log_b a})$ is the accumulative time for computing all the subproblems.
- $O(n^{\log_b a}) \times O(\sum_{i=1}^{\log_b n} (\frac{b^k}{a})^i)$ is the accumulative time for dividing problems and combining solutions.
- Three cases to consider: $\frac{b^k}{a} < 1$, $\frac{b^k}{a} = 1$, and $\frac{b^k}{a} > 1$.

Divide and Conquer Relations (cont.)

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Theorem (3.4)

The solution of the recurrence relation $T(n) = aT(n/b) + O(n^k)$, *where* a *and* b *are integer constants,* $a \geq 1$, $b \geq 2$, *and* k *is a non-negative real constant, is*

$$T(n) = \begin{cases} O(n^{\log_b a}) & \text{if } a > b^k \\ O(n^k \log n) & \text{if } a = b^k \\ O(n^k) & \text{if } a < b^k \end{cases}$$

This theorem may be slightly generalized by replacing $O(n^k)$ with some $f(n)$, but the current form is sufficient for the cases we will encounter. Due to its generality and usefulness, the theorem has conventionally been referred to as "the master theorem".

Recurrent Relations with Full History

- Example One:

$$T(n) = c + \sum_{i=1}^{n-1} T(i),$$

where c is a constant and $T(1)$ is given separately.

- $T(n) - T(n-1) = (c + \sum_{i=1}^{n-1} T(i)) - (c + \sum_{i=1}^{n-2} T(i)) = T(n-1)$; hence, $T(n) = 2T(n-1)$. (This holds only for $n \geq 3$.)
- So, we get

$$\begin{cases} T(2) = c + T(1) \\ T(n) = 2T(n-1) & \text{if } n \geq 3 \end{cases}$$

which is easier to solve.

- $T(n+1) = (c + T(1))2^{n-1}$, for $n \geq 2$.

Recurrent Relations with Full History (cont.)

- Example Two:

$$T(n) = n - 1 + \frac{2}{n}\sum_{i=1}^{n-1} T(i), (\text{for} n \geq 2). T(1) = 0.$$

- Multiply both sides of the equation with n for $T(n)$ and $(n+1)$ for $T(n+1)$.

$$nT(n) = n(n-1) + 2\sum_{i=1}^{n-1} T(i)$$
$$(n+1)T(n+1) = (n+1)n + 2\sum_{i=1}^{n} T(i)$$

- Take the difference.

$$(n+1)T(n+1) - nT(n) = (n+1)n - n(n-1) + 2T(n) = 2n + 2T(n)$$

which implies

$$T(n+1) = \frac{n+2}{n+1}T(n) + \frac{2n}{n+1}$$

Recurrent Relations with Full History (cont.)

- Further simplification.

$$T(n+1) \leq \frac{n+2}{n+1} T(n) + 2$$

- Expanding and canceling.

$$\begin{aligned}
& T(n) \\
& \leq 2 + \tfrac{n+1}{n}(2 + \tfrac{n}{n-1}(2 + \tfrac{n-1}{n-2}(\cdots(2 + \tfrac{4}{3}T(2))\cdots))) \\
& \leq 2(1 + \tfrac{n+1}{n} + \tfrac{n+1}{n}\tfrac{n}{n-1} + \tfrac{n+1}{n}\tfrac{n}{n-1}\tfrac{n-1}{n-2} + \cdots + (\tfrac{n+1}{n}\tfrac{n}{n-1}\cdots\tfrac{4}{3})) \\
& \leq 2(n+1)(\tfrac{1}{n+1} + \tfrac{1}{n} + \tfrac{1}{n-1} + \cdots + \tfrac{1}{3}) \\
& \leq 2 + 2(n+1)(\tfrac{1}{n} + \tfrac{1}{n-1} + \cdots + 1) \\
& = O(n \log n)
\end{aligned}$$

(Note: $T(1) = 0$ and $T(2) \leq 2 + \frac{3}{2}T(1) = 2$)

Useful Facts

- Bounding a summation by an integral:
  If $f(x)$ is monotonically *increasing*, then

$$\sum_{i=1}^{n} f(i) \leq \int_{1}^{n+1} f(x)dx.$$

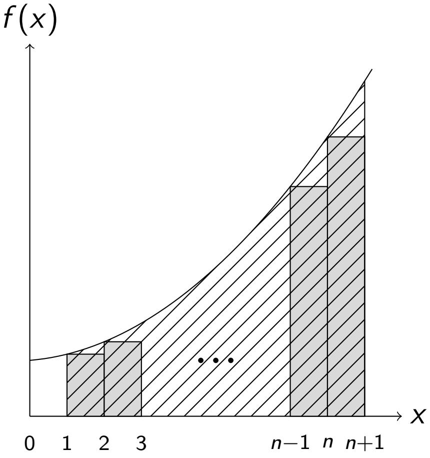
  If $f(x)$ is monotonically *decreasing*, then

$$\sum_{i=1}^{n} f(i) \leq f(1) + \int_{1}^{n} f(x)dx.$$

- Stirling's approximation

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}(1 + O(1/n)).$$

Bounding a Summation by an Integral



$$\sum_{i=1}^{n} f(i) \leq \int_{1}^{n+1} f(x)dx.$$

Useful Facts (cont.)

- Harmonic series

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + O(1/n),$$

  where $\gamma = 0.577\ldots$ is Euler's constant. So, $H_n = O(\log n)$.

- Sum of logarithms

$$\begin{aligned} \textstyle\sum_{i=1}^{n} \lfloor \log_2 i \rfloor &= (n+1)\lfloor \log_2 n \rfloor - 2^{\lfloor \log_2 n \rfloor + 1} + 2 \\ &= \Theta(n \log n). \end{aligned}$$