Algorithms 2022: Searching and Sorting

(Based on [Manber 1989])

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1 Binary Search

Searching a Sorted Sequence

Problem 1. *Let $x_1, x_2, \cdots, x_n$ be a sequence of real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$. Given a real number z, we want to find whether z appears in the sequence, and, if it does, to find an index i such that $x_i = z$.*

Idea: cut the search space in half by asking only one question.

$$\begin{cases} T(1) = O(1) \\ T(n) = T(\frac{n}{2}) + O(1), n \geq 2 \end{cases}$$

Time complexity: $O(\log n)$ (applying the master theorem with $a = 1$, $b = 2$, $k = 0$, and $b^k = 1 = a$).

Binary Search

function Find $(z, Left, Right) : integer$;
begin
    if $Left = Right$ **then**
      if $X[Left] = z$ **then** $Find := Left$
      else $Find := 0$
    else
      $Middle := \lceil \frac{Left+Right}{2} \rceil$;
      if $z < X[Middle]$ **then**
        $Find := Find(z, Left, Middle - 1)$
      else
        $Find := Find(z, Middle, Right)$
end

Algorithm Binary_Search (X, n, z);
begin
    $Position := Find(z, 1, n)$;
end

Binary Search: Alternative

function Find $(z, Left, Right) : integer$;
begin
  if $Left > Right$ **then**
    $Find := 0$
  else
    $Middle := \lceil \frac{Left+Right}{2} \rceil$;
    if $z = X[Middle]$ **then**
      $Find := Middle$
    else if $z < X[Middle]$ **then**
      $Find := Find(z, Left, Middle - 1)$
    else
      $Find := Find(z, Middle + 1, Right)$
end

How do the two algorithms compare?

1.1 Cyclically Sorted Sequence

Searching a Cyclically Sorted Sequence

Problem 2. *Given a cyclically sorted list, find the position of the minimal element in the list (we assume, for simplicity, that this position is unique).*

- Example 1:
  - $$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& 5 & 6 & 7 & 0 & 1 & 2 & 3 & 4 &] \end{array}$$
  - The 4th is the minimal element.
- Example 2:
  - $$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 &] \end{array}$$
  - The 1st is the minimal element.
- To cut the search space in half, what question should we ask?

  /* If $X[Middle] < X[Right]$, then the minimal is in the left half (including $X[Middle]$; otherwise, it is in the right half (excluding $X[Middle]$). */

Cyclic Binary Search

Algorithm Cyclic_Binary_Search (X, n);
begin
  $Position := Cyclic\_Find(1, n)$;
end

function Cyclic_Find $(Left, Right) : integer$;
begin
  if $Left = Right$ **then** $Cyclic\_Find := Left$
  else
    $Middle := \lfloor \frac{Left+Right}{2} \rfloor$;
    if $X[Middle] < X[Right]$ **then**
      $Cyclic\_Find := Cyclic\_Find(Left, Middle)$

else
$Cyclic\_Find := Cyclic\_Find(Middle+1, Right)$
end

1.2 "Fixpoints"

"Fixpoints"

Problem 3. *Given a sorted sequence of* distinct *integers* $a_1, a_2, \cdots, a_n$*, determine whether there exists an index* i *such that* $a_i = i$.

- Example 1:
  - $$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& -1 & 1 & 2 & 4 & 5 & 6 & 8 & 9 &] \end{array}$$
  - $a_4 = 4$ (there are more ...).
- Example 2:
  - $$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& -1 & 1 & 2 & 5 & 6 & 8 & 9 & 10 &] \end{array}$$
  - There is no i such that $a_i = i$.
- Again, can we cut the search space in half by asking only one question?

  /* As the numbers are distinct, they increase or decrease at least as fast as the indices (which always increase or decrease by one). If $X[Middle] < Middle$, then the fixpoint (if it exists) must be in the left half (excluding $X[Middle]$); otherwise, it must be in the right half (including $X[Middle]$). */

A Special Binary Search

function Special_Find $(Left, Right) : integer$;
begin
  if $Left = Right$ **then**
    if $A[Left] = Left$ **then** $Special\_Find := Left$
    else $Special\_Find := 0$
  else
    $Middle := \lfloor \frac{Left+Right}{2} \rfloor$;
    if $A[Middle] < Middle$ **then**
      $Special\_Find := Special\_Find(Middle+1, Right)$
    else
      $Special\_Find := Special\_Find(Left, Middle)$
end

A Special Binary Search (cont.)

Algorithm Special_Binary_Search (A, n);
begin
  $Position := Special\_Find(1, n)$;
end

1.3 Stuttering Subsequence

Stuttering Subsequence

Problem 4. *Given two sequences A ($= a_1 a_2 \cdots a_n$) and B ($= b_1 b_2 \cdots b_m$), find the maximal value of i such that B^i is a subsequence of A.*

- If $B = xyzzx$, then $B^2 = xxyyzzzzxx$, $B^3 = xxxyyyzzzzzzxxx$, etc.
- B is a subsequence of A if we can embed B inside A in the same order but with possible holes.
- For example, $B^2 = xxyyzzzzxx$ is a subsequence of $xxzzyyyyxxzzzzzzxxx$.
- If B^j is a subsequence of A, then B^i is a subsequence of A, for $1 \leq i \leq j$.
- The maximum value of i cannot exceed $\lfloor \frac{n}{m} \rfloor$ (or B^i would be longer than A).

Stuttering Subsequence (cont.)

Two ways to find the maximum i:

- Sequential search: try 1, 2, 3, etc. sequentially.

  Time complexity: $O(nj)$, where j is the maximum value of i.
- Binary search between 1 and $\lfloor \frac{n}{m} \rfloor$.

  Time complexity: $O(n \log \frac{n}{m})$.

Can binary search be applied, if the bound $\lfloor \frac{n}{m} \rfloor$ is unknown?

Think of the base case in a reversed induction.

/* Try 2^0, 2^1, 2^2, $\cdots$, 2^{k-1}, and 2^k sequentially. If the target falls between 2^{k-1} and 2^k, apply binary search within that region. */

2 Interpolation Search

Interpolation Search

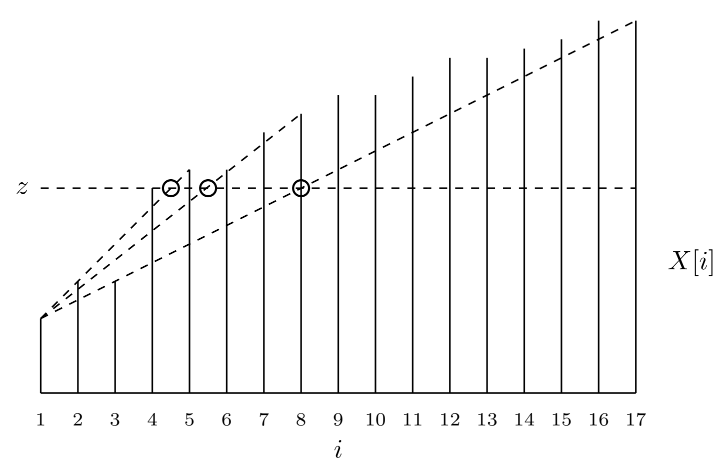
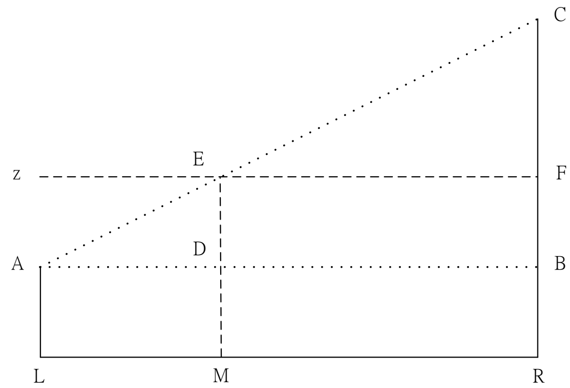


Figure: Interpolation search.

Source: redrawn from [Manber 1989, Figure 6.4].

Interpolation Search (cont.)



$$\frac{\overline{LM}}{\overline{LR}} = \frac{\overline{AD}}{\overline{AB}} = \frac{\overline{AE}}{\overline{AC}} = \frac{\overline{BF}}{\overline{BC}}, \text{so } |\overline{LM}| = \frac{|\overline{BF}|}{|\overline{BC}|} \times |\overline{LR}|$$

Interpolation Search (cont.)

function Int_Find $(z, Left, Right) : integer$;
begin
  if $X[Left] = z$ **then** $Int\_Find := Left$
  else if $Left = Right$ or $X[Left] = X[Right]$ **then**
    $Int\_Find := 0$
  else
    $Next\_Guess := \lceil Left + \frac{(z - X[Left])(Right - Left)}{X[Right] - X[Left]} \rceil$;
    if $z < X[Next\_Guess]$ **then**
      $Int\_Find := Int\_Find(z, Left, Next\_Guess - 1)$
    else
      $Int\_Find := Int\_Find(z, Next\_Guess, Right)$
end

/* $Next\_Guess - Left = |\overline{LM}| = \frac{|\overline{BF}|}{|\overline{BC}|} \times |\overline{LR}| \approx \lceil \frac{(z - X[Left])(Right - Left)}{X[Right] - X[Left]} \rceil$ */

Interpolation Search (cont.)

Algorithm Interpolation_Search (X, n, z);
begin
  if $z < X[1]$ or $z > X[n]$ **then** $Position := 0$
  else $Position := Int\_Find(z, 1, n)$;
end

3 Sorting

Sorting

Problem 5. *Given n numbers x_1, x_2, $\cdots$, x_n, arrange them in increasing order. In other words, find a sequence of distinct indices $1 \le i_1, i_2, \cdots, i_n \le n$, such that $x_{i_1} \le x_{i_2} \le \cdots \le x_{i_n}$.*

A sorting algorithm is called **in-place** if no additional work space is used besides the initial array that holds the elements.

3.1 Using Balanced Search Trees

Using Balanced Search Trees

- Balanced search trees, such as AVL trees, may be used for sorting:
  1. Create an empty tree.
  2. Insert the numbers one by one to the tree.
  3. Traverse the tree and output the numbers.
- What's the time complexity? Suppose we use an AVL tree.

3.2 Radix Sort

Radix Sort

Algorithm Straight_Radix (X, n, k);
begin
    put all elements of X in a queue GQ;
    for $i := 1$ **to** d **do**
        initialize queue $Q[i]$ to be empty
    for $i := k$ **downto** 1 **do**
        while *GQ is not empty* **do**
                pop x from GQ;
                $d :=$ the i-th digit of x;
                insert x into $Q[d]$;
        for $t := 1$ **to** d **do**
            insert $Q[t]$ into GQ;
    for $i := 1$ **to** n **do**
        pop $X[i]$ from GQ
end

Time complexity: $O(nk)$.

3.3 Merge Sort

Merge Sort

Algorithm Mergesort (X, n);
begin $M\_Sort(1, n)$ **end**

procedure M_Sort $(Left, Right)$;
begin
    if $Right - Left = 1$ **then**
        if $X[Left] > X[Right]$ **then** $swap(X[Left], X[Right])$
    else if $Left \ne Right$ **then**
            $Middle := \lceil \frac{1}{2}(Left + Right) \rceil$;
            $M\_Sort(Left, Middle - 1)$;
            $M\_Sort(Middle, Right)$;

Merge Sort (cont.)

$i := Left$; $j := Middle$; $k := 0$;
while $(i \leq Middle - 1)$ and $(j \leq Right)$ **do**
    $k := k + 1$;
    if $X[i] \leq X[j]$ **then**
        $TEMP[k] := X[i]$; $i := i + 1$
    else $TEMP[k] := X[j]$; $j := j + 1$;
if $j > Right$ **then**
    for $t := 0$ **to** $Middle - 1 - i$ **do**
        $X[Right - t] := X[Middle - 1 - t]$
for $t := 0$ **to** $k - 1$ **do**
    $X[Left + t] := TEMP[1 + t]$
end

Time complexity: $O(n \log n)$.

Merge Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	5	8	10	9	12	1	15	7	3	13	4	11	16	14
2	5	6	8	10	9	12	1	15	7	3	13	4	11	16	14
2	5	6	8	9	10	12	1	15	7	3	13	4	11	16	14
2	5	6	8	9	10	1	12	15	7	3	13	4	11	16	14
2	5	6	8	1	9	10	12	15	7	3	13	4	11	16	14
1	2	5	6	8	9	10	12	15	7	3	13	4	11	16	14
1	2	5	6	8	9	10	12	7	15	3	13	4	11	16	14
1	2	5	6	8	9	10	12	7	15	3	13	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	14	16
1	2	5	6	8	9	10	12	3	7	13	15	4	11	14	16
1	2	5	6	8	9	10	12	3	4	7	11	13	14	15	16
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Figure: An example of mergesort.
Source: redrawn from [Manber 1989, Figure 6.8].

3.4 Quick Sort

Quick Sort

Algorithm Quicksort (X, n);
begin
    $Q\_Sort(1, n)$
end

procedure Q_Sort $(Left, Right)$;
begin
    if $Left < Right$ **then**

$Partition(X, Left, Right)$;
$Q\_Sort(Left, Middle - 1)$;
$Q\_Sort(Middle + 1, Right)$
end

Time complexity: $O(n^2)$, but $O(n \log n)$ in average

Quick Sort (cont.)

Algorithm Partition$(X, Left, Right)$;
begin
$pivot := X[Left]$;
$L := Left$; $R := Right$;
while $L < R$ **do**
while $X[L] \leq pivot$ and $L \leq Right$ **do** $L := L + 1$;
while $X[R] > pivot$ and $R \geq Left$ **do** $R := R - 1$;
if $L < R$ **then** $swap(X[L], X[R])$;
$Middle := R$;
$swap(X[Left], X[Middle])$
end

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
6	2	4	5	10	9	12	1	15	7	3	13	8	11	16	14
6	2	4	5	3	9	12	1	15	7	10	13	8	11	16	14
6	2	4	5	3	1	12	9	15	7	10	13	8	11	16	14
1	2	4	5	3	6	12	9	15	7	10	13	8	11	16	14

Figure: Partition of an array around the pivot 6.
Source: redrawn from [Manber 1989, Figure 6.10].

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
1	2	4	5	3	6	12	9	15	7	10	13	8	11	16	14
1	2	4	5	3	6	12	9	15	7	10	13	8	11	16	14
1	2	4	5	3	6	12	9	15	7	10	13	8	11	16	14
1	2	3	4	5	6	12	9	15	7	10	13	8	11	16	14
1	2	3	4	5	6	8	9	11	7	10	12	13	15	16	14
1	2	3	4	5	6	7	8	11	9	10	12	13	15	16	14
1	2	3	4	5	6	7	8	10	9	11	12	13	15	16	14
1	2	3	4	5	6	7	8	9	10	11	12	13	15	16	14
1	2	3	4	5	6	7	8	9	10	11	12	13	15	16	14
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Figure: An example of quicksort.
Source: redrawn from [Manber 1989, Figure 6.12].

Average-Case Complexity of Quick Sort

- When $X[i]$ is selected (at random) as the pivot,

$$T(n) = n - 1 + T(i-1) + T(n-i), \text{ where } n \geq 2.$$

The average running time will then be

$$\begin{aligned} T(n) &= n - 1 + \tfrac{1}{n}\textstyle\sum_{i=1}^{n}(T(i-1) + T(n-i)) \\ &= n - 1 + \tfrac{1}{n}\textstyle\sum_{i=1}^{n} T(i-1) + \tfrac{1}{n}\sum_{i=1}^{n} T(n-i) \\ &= n - 1 + \tfrac{1}{n}\textstyle\sum_{j=0}^{n-1} T(j) + \tfrac{1}{n}\sum_{j=0}^{n-1} T(j) \\ &= n - 1 + \tfrac{2}{n}\textstyle\sum_{i=0}^{n-1} T(i) \end{aligned}$$

- Solving this recurrence relation with full history, $T(n) = O(n \log n)$.

3.5 Heap Sort

Heap Sort

Algorithm Heapsort (A, n);
begin
    Build_Heap(A);
    for $i := n$ **downto** 2 **do**
        swap$(A[1], A[i])$;
        Rearrange_Heap$(i-1)$
end

Time complexity: $O(n \log n)$

Heap Sort (cont.)

procedure Rearrange_Heap (k);
begin
    parent := 1;
    child := 2;
    while $child \leq k-1$ **do**
        if $A[child] < A[child+1]$ **then**
            $child := child + 1$;
        if $A[child] > A[parent]$ **then**
            swap$(A[parent], A[child])$;
            $parent := child$;
            $child := 2 * child$
        else $child := k$
end

Heap Sort (cont.)

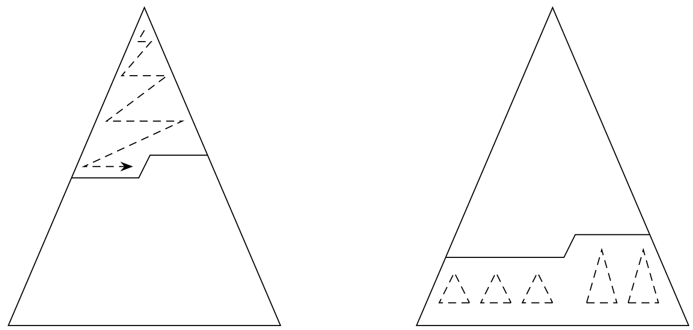


Figure: Top down and bottom up heap construction.
Source: redrawn from [Manber 1989, Figure 6.14].

How do the two approaches compare?

/* Top down: $O(n \log n)$.
Bottom up: O(sum of the heights of all nodes) $= O(n)$. Consider a full binary tree of height h. From an excercise problem in HW#2, we know that "sum of the heights of all nodes" of the tree equals $2^{h+1} - (h+2) \leq 2^{h+1} - 1 = n$. */

Building a Heap Bottom Up

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
6	2	8	5	10	9	12	14	15	7	3	13	4	11	16	1
6	2	8	5	10	9	16	14	15	7	3	13	4	11	12	1
6	2	8	5	10	13	16	14	15	7	3	9	4	11	12	1
6	2	8	5	10	13	16	14	15	7	3	9	4	11	12	1
6	2	8	15	10	13	16	14	5	7	3	9	4	11	12	1
6	2	16	15	10	13	12	14	5	7	3	9	4	11	8	1
6	15	16	14	10	13	12	2	5	7	3	9	4	11	8	1
16	15	13	14	10	9	12	2	5	7	3	6	4	11	8	1

Figure: An example of building a heap bottom up.
Source: adapted from [Manber 1989, Figure 6.15].

A Lower Bound for Sorting

- A lower bound for a particular problem is a proof that *no algorithm* can solve the problem better.
- We typically define a computation model and consider only those algorithms that fit in the model.
- **Decision trees** model computations performed by *comparison-based* algorithms.

Theorem 6 (Theorem 6.1)**.** *Every decision-tree algorithm for sorting has height $\Omega(n \log n)$.*

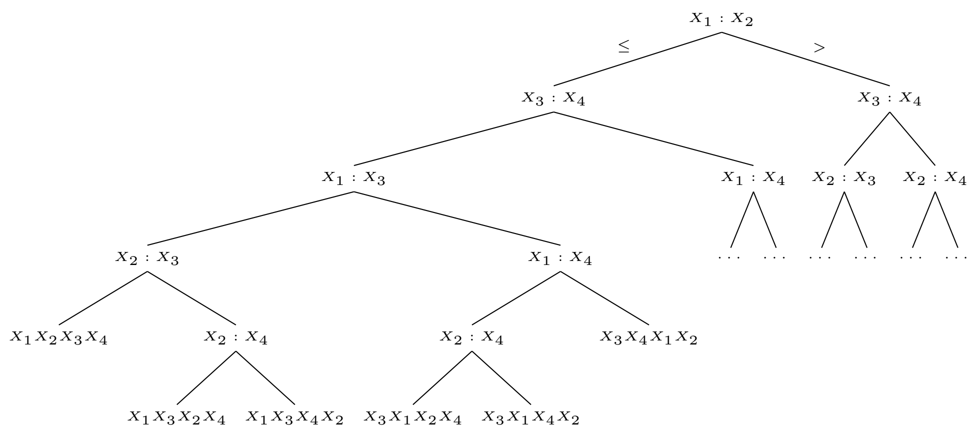
Proof idea: there must be at least $n!$ leaves in the decision tree, one for each possible outcome.

/* Recall Stirling's approximation: $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + O(1/n))$. The height of the decision tree must be at least $\log(n!)$, i.e., $\Omega(n \log n)$. */

Is the lower bound contradictory to the time complexity of radix sort?

A Lower Bound for Sorting (cont.)

A decision tree (partly shown) for the merge sort with $X_1X_2X_3X_4$ as input:



Note: in total, there should be $4! = 24$ leaves, only six of which are shown.

4 Order Statistics

Order Statistics: Minimum and Maximum

Problem 7. *Find the maximum and minimum elements in a given sequence.*

- The obvious solution requires $(n-1)+(n-2)$ $(= 2n-3)$ comparisons between elements.
- Can we do better? (Which comparisons could have been avoided?)

  /* A better algorithm: compare x_1 and x_2. Set *min* to be the smaller of the two and *max* the larger. Compare x_3 and x_4 and then compare the smaller with *min* and the larger with *max*; these take three comparisons. Update *min* and *max* accordingly. Continue until we have exhausted the sequence of numbers. Assuming n is even, the total number of comparisons $= 1 + 3 \times \frac{(n-2)}{2} = \frac{3}{2}n - 2$. */

Order Statistics: Kth-Smallest

Problem 8. *Given a sequence $S = x_1, x_2, \cdots, x_n$ of elements, and an integer k such that $1 \leq k \leq n$, find the kth-smallest element in S.*

Order Statistics: Kth-Smallest (cont.)

procedure Select $(Left, Right, k)$;
begin
    if $Left = Right$ **then**
        $Select := Left$
    else $Partition(X, Left, Right)$;
        let $Middle$ *be the output of* $Partition$;
        if $Middle - Left + 1 \geq k$ **then**
            $Select(Left, Middle, k)$
        else
            $Select(Middle + 1, Right, k - (Middle - Left + 1))$
end

Algorithm Selection (X, n, k);
begin
    if $(k < 1)$ or $(k > n)$ **then** *print "error"*
    else $S := Select(1, n, k)$
end

/* Here the formal parameter k (for rank) is made to be relative to the left bound of array indices, while $Left$, $Middle$, and $Right$ are absolute index values. */

Order Statistics: Kth-Smallest (cont.)

The nested "**if**" statement may be simplified:

procedure Select $(Left, Right, k)$;
begin
    if $Left = Right$ **then**
        $Select := Left$
    else $Partition(X, Left, Right)$;
        let $Middle$ *be the output of* $Partition$;
        if $Middle \geq k$ **then**
            $Select(Left, Middle, k)$
        else
            $Select(Middle + 1, Right, k)$
end

5 Finding a Majority

Finding a Majority

Problem 9. *Given a sequence of numbers, find the majority in the sequence or determine that none exists.*

A number is a *majority* in a sequence if it occurs more than $\frac{n}{2}$ times in the sequence.

Caution: maintaining a counter for each possible number requires $O(\log n)$ time for each access to a particular counter, which means $O(n \log n)$ time in total. Sorting the sequence to find a probable candidate also requires $O(n \log n)$ time.

Idea: compare any two numbers in the sequence. What can we conclude if they are not equal?

/* If there is a majority, it is also a majority of the other $n - 2$ numbers. However, the reverse may not be true. */

What if they are equal?

/* Keep the first number as a candidate at hand and repeat the following:
If the next number equals the candidate, we increment the count of its occurrences; otherwise, we have a pair of unequal numbers to eliminate (by decrementing the count for the candidate). When the count becomes 0 (due to elimination), we take the next number as a new candidate. */

Finding a Majority (cont.)

Algorithm Majority (X, n);
begin
    $C := X[1]$; $M := 1$;
    for $i := 2$ **to** n **do**

```
        if M = 0 then
            C := X[i];  M := 1
        else
            if C = X[i] then M := M + 1
            else M := M − 1;
```

Finding a Majority (cont.)

```
    if M = 0 then Majority := −1
    else
        Count := 0;
        for i := 1 to n do
            if X[i] = C then Count := Count + 1;
        if Count > n/2 then Majority := C
        else Majority := −1
end
```

Time complexity: $O(n)$.