Algorithms 2023: Searching and Sorting

(Based on [Manber 1989])

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1 Binary Search

Searching a Sorted Sequence

Problem 1. *Let $x_1, x_2, \cdots, x_n$ be a sequence of real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$. Given a real number z, we want to find whether z appears in the sequence, and, if it does, to find an index i such that $x_i = z$.*

Idea: cut the search space in half by asking only one question.

$$\begin{cases} T(1) = O(1) \\ T(n) = T(\frac{n}{2}) + O(1), n \geq 2 \end{cases}$$

Time complexity: $O(\log n)$ (applying the master theorem with $a = 1$, $b = 2$, $k = 0$, and $b^k = 1 = a$).

Binary Search

```
function Find (z, Left, Right) : integer;
begin
    if Left = Right then
      if X[Left] = z then Find := Left
      else Find := 0
    else
      Middle := ⌈(Left+Right)/2⌉;
      if z < X[Middle] then
        Find := Find(z, Left, Middle − 1)
      else
        Find := Find(z, Middle, Right)
end

Algorithm Binary_Search (X, n, z);
begin
    Position := Find(z, 1, n);
end
```

Binary Search: Alternative

function Find $(z, Left, Right)$: *integer*;
begin
    if $Left > Right$ **then**
        $Find := 0$
    else
        $Middle := \lceil \frac{Left+Right}{2} \rceil$;
        if $z = X[Middle]$ **then**
            $Find := Middle$
        else if $z < X[Middle]$ **then**
            $Find := Find(z, Left, Middle - 1)$
        else
            $Find := Find(z, Middle + 1, Right)$
end

How do the two algorithms compare?

/* The alternative may stop early once the target is found at *Middle*; otherwise, it spends another comparison to divide the search space. If by experience you expect to find the target almost all of the time, then consider using the alternative algorithm. */

1.1 Cyclically Sorted Sequence

Searching a Cyclically Sorted Sequence

Problem 2. *Given a cyclically sorted list, find the position of the minimal element in the list (we assume, for simplicity, that this position is unique).*

- Example 1:
  -

	1	2	3	4	5	6	7	8	
[	5	6	7	0	1	2	3	4	]

  - The 4th is the minimal element.
- Example 2:
  -

	1	2	3	4	5	6	7	8	
[	0	1	2	3	4	5	6	7	]

  - The 1st is the minimal element.
- To cut the search space in half, what question should we ask?

  /* If $X[Middle] < X[Right]$, then the minimal is in the left half (including $X[Middle]$); otherwise, it is in the right half (excluding $X[Middle]$). */

Cyclic Binary Search

Algorithm Cyclic_Binary_Search (X, n);
begin
    $Position := Cyclic\_Find(1, n)$;
end

function Cyclic_Find $(Left, Right)$: *integer*;
begin
    if $Left = Right$ **then** $Cyclic\_Find := Left$

else
Middle := $\lfloor \frac{Left+Right}{2} \rfloor$;
if $X[Middle] < X[Right]$ **then**
Cyclic_Find := *Cyclic_Find*(*Left*, *Middle*)
else
Cyclic_Find := *Cyclic_Find*(*Middle* + 1, *Right*)
end

1.2 "Fixpoints"

"Fixpoints"

Problem 3. *Given a sorted sequence of* distinct *integers* $a_1, a_2, \cdots, a_n$*, determine whether there exists an index* i *such that* $a_i = i$*.*

- Example 1:
	1	2	3	4	5	6	7	8	
[	−1	1	2	4	5	6	8	9	]
  - $a_4 = 4$ (there are more ...).
- Example 2:
	1	2	3	4	5	6	7	8	
[	−1	1	2	5	6	8	9	10	]
  - There is no i such that $a_i = i$.
- Again, can we cut the search space in half by asking only one question?

  /* As the numbers are distinct, they increase or decrease at least as fast as the indices (which always increase or decrease by one). If $X[Middle] < Middle$, then the fixpoint (if it exists) must be in the left half (excluding $X[Middle]$); otherwise, it must be in the right half (including $X[Middle]$). */

A Special Binary Search

function Special_Find (*Left*, *Right*) : *integer*;
begin
if *Left* = *Right* **then**
if $A[Left] = Left$ **then** *Special_Find* := *Left*
else *Special_Find* := 0
else
Middle := $\lfloor \frac{Left+Right}{2} \rfloor$;
if $A[Middle] < Middle$ **then**
Special_Find := *Special_Find*(*Middle* + 1, *Right*)
else
Special_Find := *Special_Find*(*Left*, *Middle*)
end

A Special Binary Search (cont.)

Algorithm Special_Binary_Search (A, n);
begin
Position := *Special_Find*(1, *n*);
end

1.3 Stuttering Subsequence

Stuttering Subsequence

Problem 4. *Given two sequences A* $(= a_1a_2\cdots a_n)$ *and B* $(= b_1b_2\cdots b_m)$*, find the maximal value of i such that* B^i *is a subsequence of A.*

- If $B = xyzzx$, then $B^2 = xxyyzzzzxx$, $B^3 = xxxyyyzzzzzzxxx$, etc.
- B is a subsequence of A if we can embed B inside A in the same order but with possible holes.
- For example, $B^2 = xxyyzzzzxx$ is a subsequence of $xxzzyyyyxxzzzzzzxxx$.
- If B^j is a subsequence of A, then B^i is a subsequence of A, for $1 \leq i \leq j$.
- The maximum value of i cannot exceed $\lfloor \frac{n}{m} \rfloor$ (or B^i would be longer than A).

Stuttering Subsequence (cont.)

Two ways to find the maximum i:

- Sequential search: try 1, 2, 3, etc. sequentially.

  Time complexity: $O(nj)$, where j is the maximum value of i.
- Binary search between 1 and $\lfloor \frac{n}{m} \rfloor$.

  Time complexity: $O(n \log \frac{n}{m})$.

Can binary search be applied, if the bound $\lfloor \frac{n}{m} \rfloor$ is unknown?

Think of the base case in a reversed induction.

/* Try 2^0, 2^1, 2^2, $\cdots$, 2^{k-1}, and 2^k sequentially. If the target falls between 2^{k-1} and 2^k, apply binary search within that region. */

2 Interpolation Search

Interpolation Search

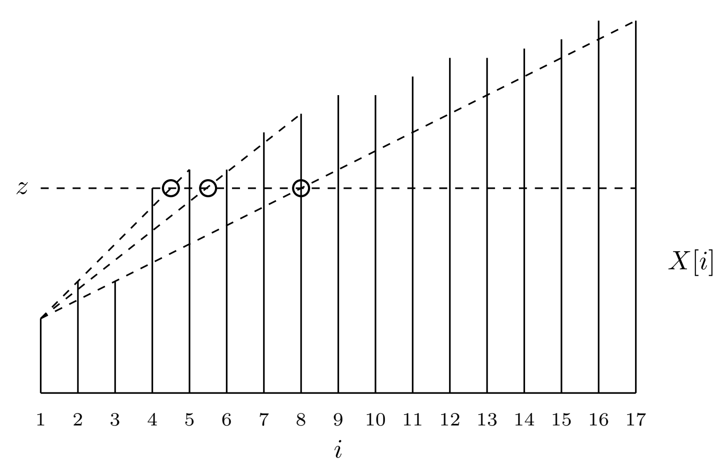
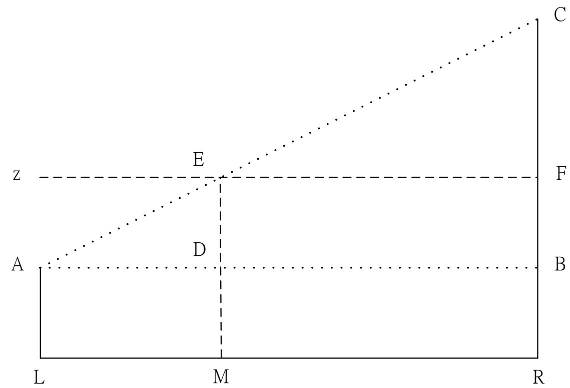


Figure: Interpolation search.

Source: redrawn from [Manber 1989, Figure 6.4].

Interpolation Search (cont.)



$$\frac{\overline{LM}}{\overline{LR}} = \frac{\overline{AD}}{\overline{AB}} = \frac{\overline{AE}}{\overline{AC}} = \frac{\overline{BF}}{\overline{BC}}, \text{so } |\overline{LM}| = \frac{|\overline{BF}|}{|\overline{BC}|} \times |\overline{LR}|$$

Interpolation Search (cont.)

function Int_Find $(z, Left, Right) : integer$;
begin
  if $X[Left] = z$ **then** $Int\_Find := Left$
  else if $Left = Right$ or $X[Left] = X[Right]$ **then**
    $Int\_Find := 0$
  else
    $Next\_Guess := \lceil Left + \frac{(z-X[Left])(Right-Left)}{X[Right]-X[Left]} \rceil$;
    if $z < X[Next\_Guess]$ **then**
      $Int\_Find := Int\_Find(z, Left, Next\_Guess - 1)$
    else
      $Int\_Find := Int\_Find(z, Next\_Guess, Right)$
end

/* $Next\_Guess - Left = |\overline{LM}| = \frac{|\overline{BF}|}{|\overline{BC}|} \times |\overline{LR}| \approx \lceil \frac{(z-X[Left])(Right-Left)}{X[Right]-X[Left]} \rceil$ */

Interpolation Search (cont.)

Algorithm Interpolation_Search (X, n, z);
begin
  if $z < X[1]$ or $z > X[n]$ **then** $Position := 0$
  else $Position := Int\_Find(z, 1, n)$;
end

3 Sorting

Sorting

Problem 5. *Given n numbers $x_1, x_2, \cdots, x_n$, arrange them in increasing order. In other words, find a sequence of distinct indices $1 \le i_1, i_2, \cdots, i_n \le n$, such that $x_{i_1} \le x_{i_2} \le \cdots \le x_{i_n}$.*

A sorting algorithm is called **in-place** if no additional work space is used besides the initial array that holds the elements.

3.1 Using Balanced Search Trees

Using Balanced Search Trees

- Balanced search trees, such as AVL trees, may be used for sorting:
  1. Create an empty tree.
  2. Insert the numbers one by one to the tree.
  3. Traverse the tree and output the numbers.
- What's the time complexity? Suppose we use an AVL tree.

  /* The time complexity is $O(n \log n)$, as there are n elements to insert and each insertion takes $O(\log n)$ time. Traversal of the tree can be done efficiently in $O(n)$ time. */

3.2 Radix Sort

Radix Sort

Algorithm Straight_Radix (X, n, k);
begin
    put all elements of X in a queue GQ;
    for $i := 1$ **to** d **do**
        initialize queue $Q[i]$ to be empty
    for $i := k$ **downto** 1 **do**
        while *GQ is not empty* **do**
            pop x from GQ;
            $d :=$ the i-th digit of x;
            insert x into $Q[d]$;
        for $t := 1$ **to** d **do**
            insert $Q[t]$ into GQ;
    for $i := 1$ **to** n **do**
        pop $X[i]$ from GQ
end

Time complexity: $O(nk)$.

3.3 Merge Sort

Merge Sort

Algorithm Mergesort (X, n);
begin *M_Sort*$(1, n)$ **end**

procedure M_Sort (*Left*, *Right*);
begin
    if *Right* $-$ *Left* $= 1$ **then**

if $X[\mathit{Left}] > X[\mathit{Right}]$ **then** $\mathit{swap}(X[\mathit{Left}], X[\mathit{Right}])$
else if $\mathit{Left} \neq \mathit{Right}$ **then**
$\mathit{Middle} := \lceil \frac{1}{2}(\mathit{Left} + \mathit{Right}) \rceil$;
$M\_Sort(\mathit{Left}, \mathit{Middle} - 1)$;
$M\_Sort(\mathit{Middle}, \mathit{Right})$;

Merge Sort (cont.)

$i := \mathit{Left}$; $j := \mathit{Middle}$; $k := 0$;
while $(i \leq \mathit{Middle} - 1)$ and $(j \leq \mathit{Right})$ **do**
$k := k + 1$;
if $X[i] \leq X[j]$ **then**
$\mathit{TEMP}[k] := X[i]$; $i := i + 1$
else $\mathit{TEMP}[k] := X[j]$; $j := j + 1$;
if $j > \mathit{Right}$ **then**
for $t := 0$ **to** $\mathit{Middle} - 1 - i$ **do**
$X[\mathit{Right} - t] := X[\mathit{Middle} - 1 - t]$
for $t := 0$ **to** $k - 1$ **do**
$X[\mathit{Left} + t] := \mathit{TEMP}[1 + t]$
end

/* In the merging stage, the while loop terminates when one of the two halves is exhausted. If the left half is exhausted ($j \leq \mathit{Right}$), the remaining elements in the right half are already in the correct positions in Array X and so nothing needs to be done. */

Time complexity: $O(n \log n)$.

/*

$$\begin{cases} T(1) = O(1) \\ T(n) = 2T(\frac{n}{2}) + O(n), n \geq 2 \end{cases}$$

Apply the master theorem with $a = 2$, $b = 2$, $k = 1$, and $b^k = 2 = a$). */

Merge Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
②	⑥	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	⑤	⑧	10	9	12	1	15	7	3	13	4	11	16	14
②	⑤	⑥	⑧	10	9	12	1	15	7	3	13	4	11	16	14
2	5	6	8	⑨	⑩	12	1	15	7	3	13	4	11	16	14
2	5	6	8	9	10	①	⑫	15	7	3	13	4	11	16	14
2	5	6	8	①	⑨	⑩	⑫	15	7	3	13	4	11	16	14
①	②	⑤	⑥	⑧	⑨	⑩	⑫	15	7	3	13	4	11	16	14
1	2	5	6	8	9	10	12	⑦	⑮	3	13	4	11	16	14
1	2	5	6	8	9	10	12	7	15	③	⑬	4	11	16	14
1	2	5	6	8	9	10	12	③	⑦	⑬	⑮	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	④	⑪	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	⑭	⑯
1	2	5	6	8	9	10	12	3	7	13	15	④	⑪	⑭	⑯
1	2	5	6	8	9	10	12	③	④	⑦	⑪	⑬	⑭	⑮	⑯
①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫	⑬	⑭	⑮	⑯

Figure: An example of mergesort.

Source: redrawn from [Manber 1989, Figure 6.8].

/* The table shows the order in which all the merges occur. However, it does not show the movements of the elements from the array to the temporary array and back. */

3.4 Quick Sort

Quick Sort

Algorithm Quicksort (X, n);
begin
  $Q\_Sort(1, n)$
end

procedure Q_Sort (*Left*, *Right*);
begin
  if *Left* < *Right* **then**
    Partition(*X*, *Left*, *Right*);
    Q_Sort(*Left*, *Middle* − 1);
    Q_Sort(*Middle* + 1, *Right*)
end

Time complexity: $O(n^2)$, but $O(n \log n)$ in average

/* The worst-case time complexity $O(n^2)$ occurs when the input array is already sorted or nearly sorted, as each partition (which is of time $O(n)$) will successively divide the array into two parts of sizes 1 and $n-2$, 1 and $n-4$, 1 and $n-6$, etc. This may be avoided by choosing the pivot more "wisely". */

Quick Sort (cont.)

Algorithm Partition(*X*, *Left*, *Right*);
begin
  pivot := *X*[*Left*];
  L := *Left*; *R* := *Right*;
  while *L* < *R* **do**
    while $X[L] \leq$ *pivot* and $L \leq$ *Right* **do** $L := L + 1$;
    while $X[R] >$ *pivot* and $R \geq$ *Left* **do** $R := R - 1$;
    if $L < R$ **then** *swap*(*X*[*L*], *X*[*R*]);
  Middle := *R*;
  swap(*X*[*Left*], *X*[*Middle*])
end

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
6	2	(4)	5	10	9	12	1	15	7	3	13	(8)	11	16	14
6	2	4	5	(3)	9	12	1	15	7	(10)	13	8	11	16	14
6	2	4	5	3	(1)	12	(9)	15	7	10	13	8	11	16	14
(1)	2	4	5	3	(6)	12	9	15	7	10	13	8	11	16	14

Figure: Partition of an array around the pivot 6.

Source: redrawn from [Manber 1989, Figure 6.10].

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
1	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	②	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	②	3	④	5	⑥	12	9	15	7	10	13	8	11	16	14
①	②	3	④	5	⑥	8	9	11	7	10	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	11	9	10	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	10	9	⑪	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	⑬	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	⑬	14	⑮	16

Figure: An example of quicksort.

Source: redrawn from [Manber 1989, Figure 6.12].

Average-Case Complexity of Quick Sort

- When $X[i]$ is selected (at random) as the pivot,

$$T(n) = n - 1 + T(i-1) + T(n-i), \text{ where } n \geq 2.$$

The average running time will then be

$$\begin{aligned} T(n) &= n - 1 + \tfrac{1}{n}\sum_{i=1}^{n}(T(i-1) + T(n-i)) \\ &= n - 1 + \tfrac{1}{n}\sum_{i=1}^{n} T(i-1) + \tfrac{1}{n}\sum_{i=1}^{n} T(n-i) \\ &= n - 1 + \tfrac{1}{n}\sum_{j=0}^{n-1} T(j) + \tfrac{1}{n}\sum_{j=0}^{n-1} T(j) \\ &= n - 1 + \tfrac{2}{n}\sum_{i=0}^{n-1} T(i) \end{aligned}$$

- Solving this recurrence relation with full history, $T(n) = O(n \log n)$.

3.5 Heap Sort

Heap Sort

Algorithm Heapsort (A, n);
begin
  Build_Heap(A);
  for $i := n$ **downto** 2 **do**
    swap$(A[1], A[i])$;
    Rearrange_Heap$(i - 1)$
end

Time complexity: $O(n \log n)$

/* The complexity is so, mainly thanks to the efficiency of *Rearrange_Heap*, which is $O(\log n)$. */

Heap Sort (cont.)

procedure Rearrange_Heap (k);
begin
    parent := 1;
    child := 2;
    while $child \leq k - 1$ **do**
        if $A[child] < A[child + 1]$ **then**
            $child := child + 1$;
        if $A[child] > A[parent]$ **then**
            $swap(A[parent], A[child])$;
            $parent := child$;
            $child := 2 * child$
        else $child := k$
end

Heap Sort (cont.)

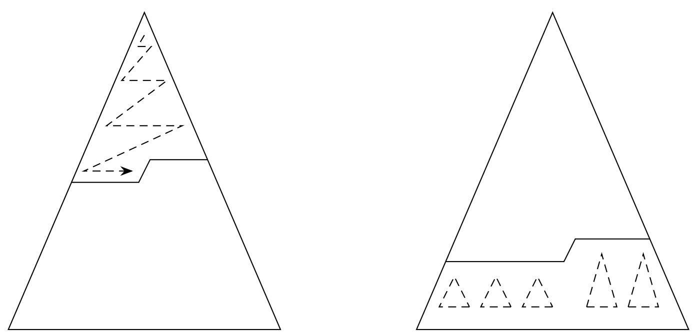


Figure: Top down and bottom up heap construction.

Source: redrawn from [Manber 1989, Figure 6.14].

How do the two approaches compare?

/* Top down: $O(n \log n)$.

Bottom up: O(sum of the heights of all nodes) $= O(n)$. Consider a full binary tree of height h. From an excercise problem in HW#2, we know that "sum of the heights of all nodes" of the tree equals $2^{h+1} - (h+2) \leq 2^{h+1} - 1 = n$. */

Building a Heap Bottom Up

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
6	2	8	5	10	9	12	(14)	15	7	3	13	4	11	16	(1)
6	2	8	5	10	9	(16)	14	15	7	3	13	4	11	(12)	1
6	2	8	5	10	(13)	16	14	15	7	3	(9)	4	11	12	1
6	2	8	5	10	13	16	14	15	7	3	9	4	11	12	1
6	2	8	(15)	10	13	16	14	(5)	7	3	9	4	11	12	1
6	2	(16)	15	10	13	(12)	14	5	7	3	9	4	11	(8)	1
6	(15)	16	(14)	10	13	12	(2)	5	7	3	9	4	11	8	1
(16)	15	(13)	14	10	(9)	12	2	5	7	3	(6)	4	11	8	1

Figure: An example of building a heap bottom up.
Source: adapted from [Manber 1989, Figure 6.15].

A Lower Bound for Sorting

- A lower bound for a particular problem is a proof that *no algorithm* can solve the problem better.
- We typically define a computation model and consider only those algorithms that fit in the model.
- **Decision trees** model computations performed by *comparison-based* algorithms.

Theorem 6 (Theorem 6.1)**.** *Every decision-tree algorithm for sorting has height* $\Omega(n \log n)$.

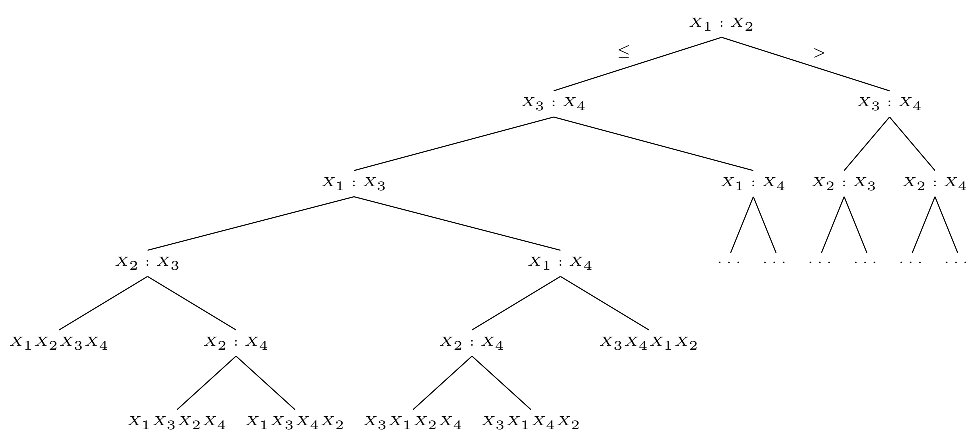
Proof idea: there must be at least $n!$ leaves in the decision tree, one for each possible outcome.

/* Recall Stirling's approximation: $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + O(1/n))$. The height of the decision tree must be at least $\log(n!)$, i.e., $\Omega(n \log n)$. */

Is the lower bound contradictory to the time complexity of radix sort?

A Lower Bound for Sorting (cont.)

A decision tree (partly shown) for the merge sort with $X_1X_2X_3X_4$ as input:



Note: in total, there should be $4! = 24$ leaves, only six of which are shown.

4 Order Statistics

Order Statistics: Minimum and Maximum

Problem 7. *Find the maximum and minimum elements in a given sequence.*

- The obvious solution requires $(n-1)+(n-2)$ $(= 2n-3)$ comparisons between elements.
- Can we do better? (Which comparisons could have been avoided?)

  /* A better algorithm: compare x_1 and x_2. Set *min* to be the smaller of the two and *max* the larger. Compare x_3 and x_4 and then compare the smaller with *min* and the larger with *max*; these take three comparisons. Update *min* and *max* accordingly. Continue until we have exhausted the sequence of numbers. Assuming n is even, the total number of comparisons $= 1 + 3 \times \frac{(n-2)}{2} = \frac{3}{2}n - 2$.

Suppose $x_1 < x_2 < x_3 < x_4$. Using the obvious solution to find the minimum and then the maximum, we would make the following six comparisons: $x_1 : x_2$, $x_1 : x_3$, and $x_1 : x_4$ and then $x_2 : x_3$ and $x_3 : x_4$. With the above algorithm, we will make just five comparisons: $x_1 : x_2$, and then $x_3 : x_4$, $x_1 : x_3$, and $x_2 : x_4$. In particular, the comparison $x_1 : x_4$ (whose result may be inferred from $x_1 : x_3$ and $x_3 : x_4$) in the obvious solution has been avoided. */

Order Statistics: Kth-Smallest

Problem 8. *Given a sequence $S = x_1, x_2, \cdots, x_n$ of elements, and an integer k such that $1 \leq k \leq n$, find the kth-smallest element in S.*

Order Statistics: Kth-Smallest (cont.)

```
procedure Select (Left, Right, k);
begin
    if Left = Right then
        Select := Left
    else Partition(X, Left, Right);
        let Middle be the output of Partition;
        if Middle − Left + 1 ≥ k then
            Select(Left, Middle, k)
        else
            Select(Middle + 1, Right, k − (Middle − Left + 1))
end
```

```
Algorithm Selection (X, n, k);
begin
    if (k < 1) or (k > n) then print "error"
    else S := Select(1, n, k)
end
```

/* Here the formal parameter k (for rank) is made to be relative to the left bound of array indices, while *Left*, *Middle*, and *Right* are absolute index values. */

Order Statistics: Kth-Smallest (cont.)

The nested "**if**" statement may be simplified:

```
procedure Select (Left, Right, k);
begin
    if Left = Right then
        Select := Left
    else Partition(X, Left, Right);
        let Middle be the output of Partition;
        if Middle ≥ k then
            Select(Left, Middle, k)
        else
            Select(Middle + 1, Right, k)
end
```

5 Finding a Majority

Finding a Majority

Problem 9. *Given a sequence of numbers, find the majority in the sequence or determine that none exists.*

A number is a *majority* in a sequence if it occurs more than $\frac{n}{2}$ times in the sequence.

Caution: maintaining a counter for each possible number requires $O(\log n)$ time for each access to a particular counter, which means $O(n \log n)$ time in total. Sorting the sequence to find a probable candidate also requires $O(n \log n)$ time.

Idea: compare any two numbers in the sequence. What can we conclude if they are not equal?

/* If there is a majority, it is also a majority of the other $n - 2$ numbers. However, the reverse may not be true. */

What if they are equal?

/* Keep the first number as a candidate at hand and repeat the following:
If the next number equals the candidate, we increment the count of its occurrences; otherwise, we have a pair of unequal numbers to eliminate (by decrementing the count for the candidate). When the count becomes 0 (due to elimination), we take the next number as a new candidate. */

Finding a Majority (cont.)

```
Algorithm Majority (X, n);
begin
    C := X[1];  M := 1;
    for i := 2 to n do
        if M = 0 then
            C := X[i];  M := 1
        else
            if C = X[i] then M := M + 1
            else M := M − 1;
```

Finding a Majority (cont.)

```
    if M = 0 then Majority := −1
    else
        Count := 0;
        for i := 1 to n do
            if X[i] = C then Count := Count + 1;
        if Count > n/2 then Majority := C
        else Majority := −1
end
```

Time complexity: $O(n)$.