Basic Graph Algorithms

(Based on [Manber 1989])

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The Königsberg Bridges Problem

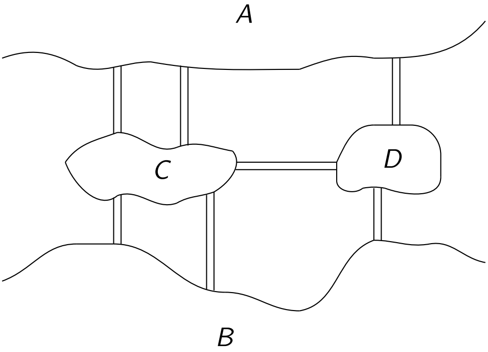

Figure: The Königsberg bridges problem.

Source: redrawn from [Manber 1989, Figure 7.1].

Can one start from one of the lands, cross every bridge exactly once, and return to the origin?

The Königsberg Bridges Problem (cont.)

An abstract model is more convenient to work with:

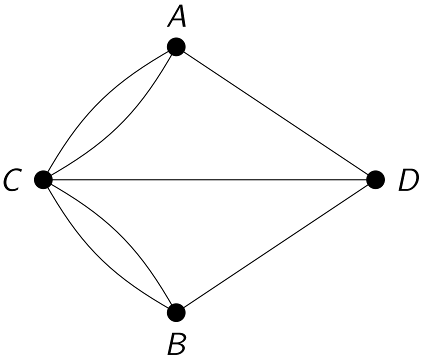

Figure: The graph corresponding to the Königsberg bridges problem.

Source: redrawn from [Manber 1989, Figure 7.2].

Graphs

- A graph consists of a set of vertices (or nodes) and a set of edges (or links, each normally connecting two vertices).
- A graph is commonly denoted as $G(V, E)$, where
  - G is the name of the graph,
  - V is the set of vertices, and
  - E is the set of edges.

Note: we assume that you have learned from a course on Data Structures the basics of graph theory and the representation of a graph by an adjacency matrix or incidence list.

Graphs (cont.)

- Undirected vs. Directed Graph
- Simple Graph vs. Multigraph
- Path, Simple Path, Trail
- Cycle, Simple Cycle, Circuit
- Degree, In-Degree, Out-Degree
- Connected Graph, Connected Components
- Tree, Forest
- Subgraph, Induced Subgraph
- Spanning Tree, Spanning Forest
- Weighted Graph

Modeling with Graphs

- Reachability
  - Finding program errors
  - Solving sliding tile puzzles
- Shortest Paths
  - Finding the fastest route to a place
  - Routing messages in networks
- Graph Coloring
  - Coloring maps
  - Scheduling classes

Eulerian Graphs

Problem

Given an undirected connected graph $G = (V, E)$ such that all the vertices have even degrees, find a circuit P such that each edge of E appears in P exactly once.

The circuit P in the problem statement is called an *Eulerian circuit*.

Theorem

An undirected connected graph has an Eulerian circuit if and only if all of its vertices have even degrees.

Depth-First Search

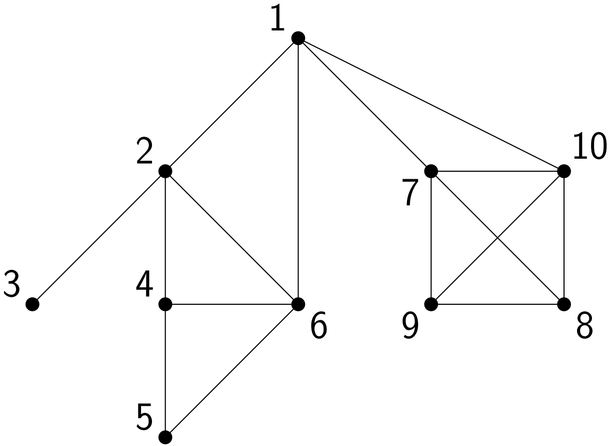

Figure: A DFS for an undirected graph.

Source: redrawn from [Manber 1989, Figure 7.4].

Depth-First Search (cont.)

```
Algorithm Depth_First_Search($G, v$);
begin
    mark $v$;
    perform preWORK on $v$;
    for all edges $(v, w)$ do
        if $w$ is unmarked then
            Depth_First_Search($G, w$);
        perform postWORK for $(v, w)$
end
```

Depth-First Search (cont.)

```
Algorithm Refined_DFS($G, v$);
begin
    mark $v$;
    perform preWORK on $v$;
    for all edges $(v, w)$ do
        if $w$ is unmarked then
            Refined_DFS($G, w$);
        perform postWORK for $(v, w)$;
    perform postWORK_II on $v$
end
```

A “Metaphor” of DFS

> *Space: the final frontier. These are the voyages of the starship Enterprise. Its five-year mission: to explore strange new worlds. To seek out new life and new civilizations. To boldly go where no man/one has gone before!*
>
> – Captain James T. Kirk, *Star Trek*

Connected Components

```
Algorithm Connected_Components(G);
begin
    Component_Number := 1;
    while there is an unmarked vertex v do
        Depth_First_Search(G, v)
        (preWORK:
            v.Component := Component_Number);
        Component_Number := Component_Number + 1
end
```

Connected Components

```
Algorithm Connected_Components(G);
begin
    Component_Number := 1;
    while there is an unmarked vertex v do
        Depth_First_Search(G, v)
        (preWORK:
            v.Component := Component_Number);
        Component_Number := Component_Number + 1
end
```

Time complexity:

Connected Components

```
Algorithm Connected_Components(G);
begin
    Component_Number := 1;
    while there is an unmarked vertex v do
        Depth_First_Search(G, v)
        (preWORK:
            v.Component := Component_Number);
        Component_Number := Component_Number + 1
end
```

Time complexity: $O(|E| + |V|)$.

DFS Numbers

```
**Algorithm DFS_Numbering**$(G, v)$;
begin
    $DFS\_Number := 1$;
    $Depth\_First\_Search(G, v)$
    (preWORK:
        $v.DFS := DFS\_Number$;
        $DFS\_Number := DFS\_Number + 1$)
end
```

DFS Numbers

```
Algorithm DFS_Numbering(G, v);
begin
    DFS_Number := 1;
    Depth_First_Search(G, v)
    (preWORK:
        v.DFS := DFS_Number;
        DFS_Number := DFS_Number + 1)
end
```

Time complexity: $O(|E|)$ (assuming the input graph is connected).

The DFS Tree

```
Algorithm Build_DFS_Tree($G, v$);
begin
    Depth_First_Search($G, v$)
    (postWORK:
        if $w$ was unmarked then
            add the edge $(v, w)$ to $T$);
end
```

The DFS Tree (cont.)

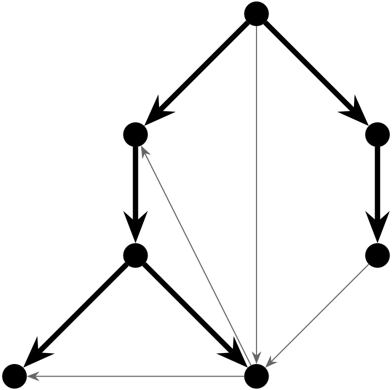

Figure: A DFS tree for a directed graph.

Source: redrawn from [Manber 1989, Figure 7.9].

The DFS Tree (cont.)

Lemma (7.2)

For an undirected graph $G = (V, E)$, every edge $e \in E$ either belongs to the DFS tree T, or connects two vertices of G, one of which is the ancestor of the other in T.

For undirected graphs, DFS avoids cross edges.

Lemma (7.3)

For a directed graph $G = (V, E)$, if (v, w) is an edge in E such that $v.DFS\_Number < w.DFS\_Number$, then w is a descendant of v in the DFS tree T.

For directed graphs, cross edges must go "from right to left".

Directed Cycles

Problem

Given a directed graph $G = (V, E)$, determine whether it contains a (directed) cycle.

Lemma (7.4)

G contains a directed cycle if and only if G contains a back edge (relative to the DFS tree).

Directed Cycles (cont.)

```
Algorithm Find_a_Cycle(G);
begin
    Depth_First_Search(G, v) /* arbitrary v */
    (preWORK:
        v.on_the_path := true;
     postWORK:
        if w.on_the_path then
            Find_a_Cycle := true;
            halt;
        if w is the last vertex on v's list then
            v.on_the_path := false;)
end
```

Directed Cycles (cont.)

```
Algorithm Refined_Find_a_Cycle($G$);
begin
    Refined_DFS($G, v$) /* arbitrary $v$ */
    (preWORK:
        $v.on\_the\_path := true$;
     postWORK:
        if $w.on\_the\_path$ then
            $Refined\_Find\_a\_Cycle := true$;
            halt;
     postWORK_II:
        $v.on\_the\_path := false$)
end
```

Breadth-First Search

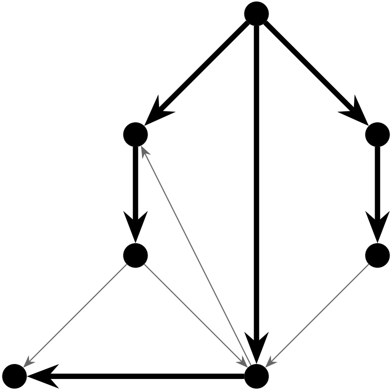

Figure: A BFS tree for a directed graph.

Source: redrawn from [Manber 1989, Figure 7.12].

Breadth-First Search (cont.)

```
Algorithm Breadth_First_Search($G, v$);
begin
    mark $v$;
    put $v$ in a queue;
    while the queue is not empty do
        remove vertex $w$ from the queue;
        perform preWORK on $w$;
        for all edges $(w, x)$ with $x$ unmarked do
            mark $x$;
            add $(w, x)$ to the $BFS$ tree $T$;
            put $x$ in the queue
end
```

Breadth-First Search (cont.)

Lemma (7.5)

If an edge (u, w) belongs to a BFS tree such that u is a parent of w, then u has the minimal BFS number among vertices with edges leading to w.

Lemma (7.6)

For each vertex w, the path from the root to w in T is a shortest path from the root to w in G.

Lemma (7.7)

If an edge (v, w) in E does not belong to T and w is on a larger level, then the level numbers of w and v differ by at most 1.

Breadth-First Search (cont.)

```
Algorithm Simple_BFS($G, v$);
begin
    put $v$ in $Queue$;
    while $Queue$ is not empty do
        remove vertex $w$ from $Queue$;
        if $w$ is unmarked then
            mark $w$;
            perform preWORK on $w$;
            for all edges $(w, x)$ with $x$ unmarked do
                put $x$ in $Queue$
end
```

Breadth-First Search (cont.)

```
Algorithm Simple_Nonrecursive_DFS($G, v$);
begin
    push $v$ to Stack;
    while Stack is not empty do
        pop vertex $w$ from Stack;
        if $w$ is unmarked then
            mark $w$;
            perform preWORK on $w$;
            for all edges $(w, x)$ with $x$ unmarked do
                push $x$ to Stack
end
```

Topological Sorting

Problem

Given a directed acyclic graph $G = (V, E)$ with n vertices, label the vertices from 1 to n such that, if v is labeled k, then all vertices that can be reached from v by a directed path are labeled with labels $> k$.

Lemma (7.8)

A directed acyclic graph always contains a vertex with indegree 0.

Topological Sorting (cont.)

```
Algorithm Topological_Sorting($G$);
    initialize $v.indegree$ for all vertices; /* by DFS */
    $G\_label := 0$;
    for $i := 1$ to $n$ do
        if $v_i.indegree = 0$ then put $v_i$ in Queue;
    repeat
        remove vertex $v$ from Queue;
        $G\_label := G\_label + 1$;
        $v.label := G\_label$;
        for all edges $(v, w)$ do
            $w.indegree := w.indegree - 1$;
            if $w.indegree = 0$ then put $w$ in Queue
    until Queue is empty
```

Single-Source Shortest Paths

Problem

Given a directed graph $G = (V, E)$ *and a vertex* v*, find shortest paths from* v *to all other vertices of* G*.*

Shorted Paths: The Acyclic Case

```
Algorithm Acyclic_Shortest_Paths($G, v, n$);
{Initially, $w.SP = \infty$, for every node $w$.}
{A topological sort has been performed on $G$, ...}
begin
    let $z$ be the vertex labeled $n$;
    if $z \neq v$ then
        Acyclic_Shortest_Paths($G - z, v, n - 1$);
        for all $w$ such that $(w, z) \in E$ do
            if $w.SP + length(w, z) < z.SP$ then
                $z.SP := w.SP + length(w, z)$
    else $v.SP := 0$
end
```

The Acyclic Case (cont.)

```
**Algorithm Imp_Acyclic_Shortest_Paths**$(G, v)$;
    for all vertices $w$ do $w.SP := \infty$;
    initialize $v.indegree$ for all vertices;
    for $i := 1$ to $n$ do
        if $v_i.indegree = 0$ then put $v_i$ in $Queue$;
    $v.SP := 0$;
    repeat
        remove vertex $w$ from $Queue$;
        for all edges $(w, z)$ do
            if $w.SP + length(w, z) < z.SP$ then
                $z.SP := w.SP + length(w, z)$;
            $z.indegree := z.indegree - 1$;
            if $z.indegree = 0$ then put $z$ in $Queue$
    until $Queue$ is empty
```

Shortest Paths: The General Case

```
Algorithm Single_Source_Shortest_Paths($G, v$);
// Dijkstra's algorithm
begin
    for all vertices $w$ do
        $w.mark := false$;
        $w.SP := \infty$;
    $v.SP := 0$;
    while there exists an unmarked vertex do
        let $w$ be an unmarked vertex s.t. $w.SP$ is minimal;
        $w.mark := true$;
        for all edges $(w, z)$ such that $z$ is unmarked do
            if $w.SP + length(w, z) < z.SP$ then
                $z.SP := w.SP + length(w, z)$
end
```

Shortest Paths: The General Case

```
Algorithm Single_Source_Shortest_Paths($G, v$);
// Dijkstra's algorithm
begin
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    $v.SP := 0$;
    while there exists an unmarked vertex do
        let $w$ be an unmarked vertex s.t. $w.SP$ is minimal;
        $w.mark := true$;
        for all edges $(w, z)$ such that $z$ is unmarked do
            if $w.SP + length(w, z) < z.SP$ then
                $z.SP := w.SP + length(w, z)$
end
```

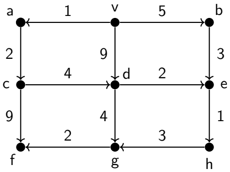
Time complexity:

Shortest Paths: The General Case

```
Algorithm Single_Source_Shortest_Paths($G, v$);
// Dijkstra's algorithm
begin
    for all vertices $w$ do
        $w.mark := false$;
        $w.SP := \infty$;
    $v.SP := 0$;
    while there exists an unmarked vertex do
        let $w$ be an unmarked vertex s.t. $w.SP$ is minimal;
        $w.mark := true$;
        for all edges $(w, z)$ such that $z$ is unmarked do
            if $w.SP + length(w, z) < z.SP$ then
                $z.SP := w.SP + length(w, z)$
end
```

Time complexity: $O((|E| + |V|) \log |V|)$ (using a min heap).

The General Case (cont.)



	v	a	b	c	d	e	f	g	h
a	0	1	5	∞	9	∞	∞	∞	∞
c	0	①	5	3	9	∞	∞	∞	∞
b	0	①	5	③	7	∞	12	∞	∞
d	0	①	⑤	③	7	8	12	∞	∞
e	0	①	⑤	③	⑦	8	12	∞	∞
h	0	①	⑤	③	⑦	⑧	12	11	9
g	0	①	⑤	③	⑦	⑧	12	11	⑨
f	0	①	⑤	③	⑦	⑧	12	⑪	⑨

Figure: An example of the single-source shortest-paths algorithm.

Source: redrawn from [Manber 1989, Figure 7.18].

Minimum-Weight Spanning Trees

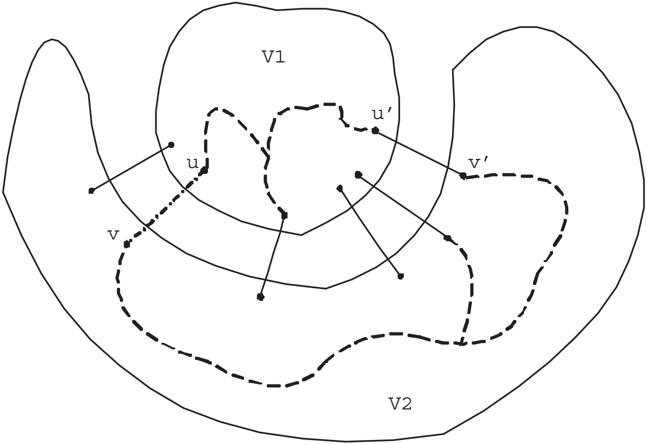
Problem

Given an undirected connected weighted graph $G = (V, E)$, find a spanning tree T of G of minimum weight.

Theorem

Let V_1 and V_2 be a partition of V and $E(V_1, V_2)$ be the set of edges connecting nodes in V_1 to nodes in V_2. The edge with the minimum weight in $E(V_1, V_2)$ must be in the minimum-cost spanning tree of G.

Minimum-Weight Spanning Trees (cont.)



If $cost(u, v)$ is the smallest among $E(V_1, V_2)$, then $\{u, v\}$ must be in the minimum spanning tree.

Minimum-Weight Spanning Trees (cont.)

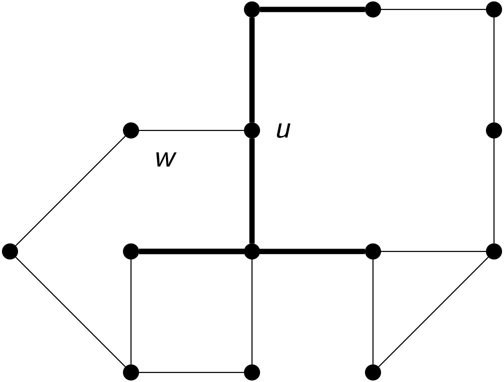


Figure: Finding the next edge of the MCST.

Source: redrawn from [Manber 1989, Figure 7.19].

Minimum-Weight Spanning Trees (cont.)

```
**Algorithm MST**($G$);
// A variant of Prim's algorithm
begin
    initially $T$ is the empty set;
    for all vertices $w$ do
        $w.mark := false$;  $w.cost := \infty$;
    let $(x, y)$ be a minimum cost edge in $G$;
    $x.mark := true$;
    for all edges $(x, z)$ do
        $z.edge := (x, z)$;  $z.cost := cost(x, z)$;
```

Minimum-Weight Spanning Trees (cont.)

```
    while there exists an unmarked vertex do
        let $w$ be an unmarked vertex with minimal $w.cost$;
        if $w.cost = \infty$ then
            print "G is not connected";  halt
        else
            $w.mark := true$;
            add $w.edge$ to $T$;
            for all edges $(w, z)$ do
                if not $z.mark$ then
                    if $cost(w, z) < z.cost$ then
                        $z.edge := (w, z)$;  $z.cost := cost(w, z)$
end
```

Minimum-Weight Spanning Trees (cont.)

```
Algorithm Another_MST(G);
// Prim's algorithm
begin
    initially T is the empty set;
    for all vertices w do
        w.mark := false;  w.cost := ∞;
    x.mark := true; /* x is an arbitrary vertex */
    for all edges (x, z) do
        z.edge := (x, z);  z.cost := cost(x, z);
```

Minimum-Weight Spanning Trees (cont.)

```
    while there exists an unmarked vertex do
        let $w$ be an unmarked vertex with minimal $w.cost$;
        if $w.cost = \infty$ then
            print "G is not connected"; halt
        else
            $w.mark := true$;
            add $w.edge$ to $T$;
            for all edges $(w, z)$ do
                if not $z.mark$ then
                    if $cost(w, z) < z.cost$ then
                        $z.edge := (w, z)$;
                        $z.cost := cost(w, z)$
end
```

Minimum-Weight Spanning Trees (cont.)

```
    while there exists an unmarked vertex do
        let $w$ be an unmarked vertex with minimal $w.cost$;
        if $w.cost = \infty$ then
            print "G is not connected"; halt
        else
            $w.mark := true$;
            add $w.edge$ to $T$;
            for all edges $(w, z)$ do
                if not $z.mark$ then
                    if $cost(w, z) < z.cost$ then
                        $z.edge := (w, z)$;
                        $z.cost := cost(w, z)$
end
```

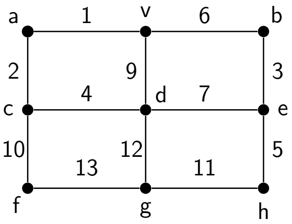
Time complexity:

Minimum-Weight Spanning Trees (cont.)

```
    while there exists an unmarked vertex do
        let $w$ be an unmarked vertex with minimal $w.cost$;
        if $w.cost = \infty$ then
            print "G is not connected"; halt
        else
            $w.mark := true$;
            add $w.edge$ to $T$;
            for all edges $(w, z)$ do
                if not $z.mark$ then
                    if $cost(w, z) < z.cost$ then
                        $z.edge := (w, z)$;
                        $z.cost := cost(w, z)$
end
```

Time complexity: same as that of Dijkstra's algorithm.

Minimum-Weight Spanning Trees (cont.)



	v	a	b	c	d	e	f	g	h
v	-	v(1)	v(6)	∞	v(9)	∞	∞	∞	∞
a	-	-	v(6)	*a*(2)	v(9)	∞	∞	∞	∞
c	-	-	v(6)	-	c(4)	∞	*c*(10)	∞	∞
d	-	-	v(6)	-	-	d(7)	*c*(10)	d(12)	∞
b	-	-	-	-	-	b(3)	*c*(10)	d(12)	∞
e	-	-	-	-	-	-	*c*(10)	d(12)	e(5)
h	-	-	-	-	-	-	*c*(10)	h(11)	-
f	-	-	-	-	-	-	—	h(11)	-
g	-	-	-	-	-	-	—	-	-

Figure: An example of the minimum-cost spanning-tree algorithm.

Source: redrawn from [Manber 1989, Figure 7.21].

All Shortest Paths

Problem

Given a weighted graph $G = (V, E)$ *(directed or undirected) with nonnegative weights, find the minimum-length paths between all pairs of vertices.*

All Shortest Paths

Problem

Given a weighted graph $G = (V, E)$ (directed or undirected) with nonnegative weights, find the minimum-length paths between all pairs of vertices.

Basic ideas (of Floyd's algorithm):

- Introduce the notion of a k-path, where the largest number of the intermediate vertices is k.
- Induct over the sequence of numbers of the vertices.
- The best m-path from u to v is the best $(< m)$-path from u to m combined with the best $(< m)$-path from m to v.

Floyd's Algorithm

```
Algorithm All_Pairs_Shortest_Paths(W);
begin
    {initialization}
    for $i := 1$ to $n$ do
        for $j := 1$ to $n$ do
            if $(i, j) \in E$ then $W[i, j] := length(i, j)$
            else $W[i, j] := \infty$;
    for $i := 1$ to $n$ do $W[i, i] := 0$;

    for $m := 1$ to $n$ do  {the induction sequence}
        for $x := 1$ to $n$ do
            for $y := 1$ to $n$ do
                if $W[x, m] + W[m, y] < W[x, y]$ then
                    $W[x, y] := W[x, m] + W[m, y]$
end
```

Transitive Closure

Problem

Given a directed graph $G = (V, E)$, find its transitive closure.

```
Algorithm Transitive_Closure($A$);
begin
    {initialization omitted}
    for $m := 1$ to $n$ do
        for $x := 1$ to $n$ do
            for $y := 1$ to $n$ do
                if $A[x, m]$ and $A[m, y]$ then
                    $A[x, y] := true$
end
```

Transitive Closure (cont.)

```
Algorithm Improved_Transitive_Closure($A$);
begin
    {initialization omitted}
    for $m$ := 1 to $n$ do
        for $x$ := 1 to $n$ do
            if $A[x, m]$ then
                for $y$ := 1 to $n$ do
                    if $A[m, y]$ then
                        $A[x, y]$ := true
end
```