Homework 3

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Question 1

1. (3.5) For each of the following pairs of functions, determine whether $f(n) = O(g(n))$ and/or $f(n) = \Omega(g(n))$. Justify your answers.

	$f(n)$	$g(n)$
(a)	$\frac{n}{\log n}$	$(\log n)^2$
(b)	$n^3 2^n$	3^n

Question 1

$f(n) = O(g(n))$:
$\exists c\ , N > 0\ s.t.\ f(n) \leq c \cdot g(n)$ holds $\forall n \geq N$.

$f(n) = \Omega(g(n))$:
$\exists c\ , N > 0\ s.t.\ f(n) \geq c \cdot g(n)$ holds $\forall n \geq N$.

$f(n) = o(g(n))$:
$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

$\Rightarrow$ if $f(n) = o(g(n))$, then $f(n) = O(g(n))$ and $f(n) \neq \Omega(g(n))$.

$\Rightarrow$ if $g(n) = o(f(n))$, then $f(n) = \Omega(g(n))$ and $f(n) \neq O(g(n))$.

Question 1 (a)

$f(n) = \frac{n}{\log n}$, $g(n) = (\log n)^2$:

$$\begin{aligned}
\lim_{n \to \infty} \frac{g(n)}{f(n)} &= \lim_{n \to \infty} \frac{(\log n)^3}{n} \\
&\overset{\text{L'H}}{=} \lim_{n \to \infty} \frac{3(\log n)^2}{n \ln 10} \\
&\overset{\text{L'H}}{=} \lim_{n \to \infty} \frac{6 \log n}{n (\ln 10)^2} \\
&\overset{\text{L'H}}{=} \lim_{n \to \infty} \frac{6}{n (\ln 10)^3} \\
&= 0
\end{aligned}$$

Question 1 (a)

Since $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$, $g(n) = o(f(n))$,

which implies $f(n) = \Omega(g(n))$ and $f(n) \neq O(g(n))$.

Question 1 (b)

$f(n) = n^3 2^n$, $g(n) = 3^n$:

$$\begin{aligned}\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^3 2^n}{3^n} = \lim_{n\to\infty} \frac{n^3}{(\frac{3}{2})^n} \\ \{\text{apply L'Hôpital's rule 3 times}\} \\ = \lim_{n\to\infty} \frac{6}{(\ln \frac{3}{2})^3 (\frac{3}{2})^n} \\ = 0\end{aligned}$$

Question 1 (b)

Since $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, $f(n) = o(g(n))$,

which implies $f(n) = O(g(n))$ and $f(n) \neq \Omega(g(n))$.

Question 2

2. Suppose f is a strictly increasing function that maps every positive integer to another positive integer, i.e., if $1 \leq n_1 < n_2$, then $1 \leq f(n_1) < f(n_2)$, and $f(n) = O(g(n))$ for some other function g. Is it true that $\log f(n) = O(\log g(n))$? Please justify your answer. How about $2^{f(n)} = O(2^{g(n)})$? Is it true?

Question 2

(big-o) $f(n) = O(g(n))$:
$\exists c, N > 0$ $s.t.$ $f(n) \leq c \cdot g(n)$ holds $\forall n \geq N$.

(1) $\log f(n) \stackrel{?}{=} O(\log g(n))$

$$\begin{aligned} \log f(n) \leq \log(c \cdot g(n)) &= \log(c) + \log g(n) \\ &= \log g(n) \cdot (1 + \frac{\log(c)}{\log g(n)}) \\ &\leq \log g(n) \cdot (1 + \frac{\log(c)}{\log g(N)}) \\ &\leq \log g(n) \cdot c', \text{where } c' > 0. \end{aligned}$$

s.t. $\log f(n) \leq c' \cdot \log g(n)$ holds $\forall n \geq N$.

$\Rightarrow \log f(n) = O(\log g(n))$.

Question 2

(2) $2^{f(n)} \stackrel{?}{=} O(2^{g(n)})$

We can only get $2^{f(n)} \le 2^{c \cdot g(n)} = (2^{g(n)})^c$

and it doesn't imply $2^{f(n)} = O(2^{g(n)})$.
The hypothesis $2^{f(n)} = O(2^{g(n)})$ can simply be rejected with

a counter example: $f(n) = 2\log_2(n)$, $g(n) = \log_2(n)$.

$f(n) = O(\log(n)) = O(g(n))$

$2^{f(n)} = 2^{2\log_2(n)} = n^2 = O(n^2)$

$2^{g(n)} = 2^{\log_2(n)} = n = O(n)$

$O(n^2) \notin O(n)$

$\Rightarrow 2^{f(n)} \neq O(2^{g(n)})$

Question 3

3. Solve the following recurrence relation using *generating functions*. This is a very simple recurrence relation, but for the purpose of practicing you must use generating functions in your solution.

$$\begin{cases} T(1) = 1 \\ T(2) = 3 \\ T(n) = T(n-1) + 2\,T(n-2), \quad n \geq 3 \end{cases}$$

Question 3

Let $T_n = T(n)$, $G(z) = T_1 + T_2 z + T_3 z^2 + \cdots + T_n z^{n-1} + \cdots$.

$$\begin{aligned}
G(z) &= T_1 + T_2 z + T_3 z^2 + \cdots + T_{n-1} z^n + \cdots \\
zG(z) &= \qquad T_1 z + T_2 z^2 + \cdots + T_{n-2} z^n + \cdots \\
2z^2 G(z) &= \qquad\qquad 2T_1 z^2 + \cdots + 2T_{n-3} z^n + \cdots \\
\hline
(1 - z - 2z^2)G(z) &= T_1 + (T_2 - T_1)z - (T_3 - T_2 - 2T_1)z^2 \\
&= 1 + 2z + 0 \\
&= 1 + 2z
\end{aligned}$$

$$G(z) = \frac{1 + 2z}{1 - z - 2z^2}$$

Question 3

$$
\begin{aligned}
G(z) &= \frac{1+2z}{1-z-2z^2} \\
&= \frac{1+2z}{(1-2z)(1+z)} \\
&= \frac{\frac{4}{3}}{1-2z} - \frac{\frac{1}{3}}{1+z} \\
&= T_1 + T_2 z + \cdots + [\frac{4}{3}(2^{n-1}) - \frac{1}{3}(-1)^{n-1}]z^n + \cdots
\end{aligned}
$$

$T(n) = \frac{4}{3}(2^{n-1}) - \frac{1}{3}(-1)^{n-1}$

Question 4

4. (3.26) Find the asymptotic behavior of the function $T(n)$ defined by the recurrence relation

$$\begin{cases} T(1) = 1 \\ T(n) = T(n/2) + \sqrt{n}, \quad n \geq 2. \end{cases}$$

You can consider only values of n that are powers of 2.

Question 4

We first observe that since $T(n) = T(n/2) + \sqrt{n}$ when $n \geq 2$, and for some $k \in \mathbb{N}$, we have

$$T(2) = T(1) + \sqrt{2}$$
$$\cdots$$
$$T(2^{k-1}) = T(2^{k-2}) + \sqrt{2^{k-1}}$$
$$T(2^k) = T(2^{k-1}) + \sqrt{2^k}$$
$$\cdots$$

, that is, when $n = 2^k$,

$$T(n) = T(1) + \sqrt{2} + \cdots + \sqrt{2^{k-1}} + \sqrt{2^k}.$$

Question 4

(Cont.)

$$
\begin{aligned}
T(n) = & \ T(1) + \sqrt{2} + \cdots + \sqrt{2^{k-1}} + \sqrt{2^k} \\
& = 1 + \sqrt{\frac{2^k}{2^{k-1}}} + \cdots + \sqrt{\frac{2^k}{2^1}} + \sqrt{\frac{2^k}{2^0}} \\
& = 1 + \sqrt{\frac{n}{2^{k-1}}} + \cdots + \sqrt{\frac{n}{2^1}} + \sqrt{\frac{n}{2^0}} \\
& = 1 + \sqrt{n} \cdot (1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{2^{k-1}}}) \\
& \leq 1 + \sqrt{n} \cdot (1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{2^{k-1}}} + \cdots).
\end{aligned}
$$

Question 4

By the generating function:

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots$$

, we have

$$T(n) \leq 1 + \sqrt{n} \cdot \frac{1}{1 - \frac{1}{\sqrt{2}}}.$$

Note that the constant 1 and $\frac{1}{1-\frac{1}{\sqrt{2}}}$ are not relevant in asymptotic notation, thus, the asymptotic behavior of $T(n)$ is $O(\sqrt{n})$.

Question 5

5. (3.30) Use Equation 1, shown below, to prove that $S(n) = \sum_{i=1}^{n} \lceil \log_2(n/i) \rceil$ satisfies $S(n) = O(n)$.

Bounding a summation by an integral
If $f(x)$ is a monotonically increasing continuous function, then

$$\sum_{i=1}^{n} f(i) \leq \int_{x=1}^{x=n+1} f(x)dx. \tag{1}$$

Question 5

$S(n) = \sum_{i=1}^{n} \lceil \log_2(\frac{n}{i}) \rceil \le \sum_{i=1}^{n} (\log_2(\frac{n}{i}) + 1) = \sum_{i=1}^{n} \log_2(\frac{n}{i}) + n$

Let $f(x) = \log_2(\frac{n}{n-x+1}) \Rightarrow \sum_{x=1}^{n} f(x) \le \int_1^{n+1} f(x)\, dx$

Now, we calculate $\int_1^{n+1} f(x)\, dx$ to assess our S(n):

$$\begin{aligned}
\int_1^{n+1} f(x)\, dx &= \int_1^{n+1} \log_2(n) - \log_2(n-x+1)\, dx \\
&= n \log_2 n - \int_1^{n+1} \log_2(n-x+1)\, dx \\
&\{\text{Substitute } n-x+1 \text{ by } u\} \\
&= n \log_2 n - \frac{1}{\ln 2} [-(u)\ln(u) + (u)] \Big|_n^0
\end{aligned}$$

Question 5

$$\int_1^{n+1} f(x)\,dx = n\log_2 n - \frac{1}{\ln 2}(n\ln(n) - n)$$
$$= n\log_2 n - n\log_2 n + \frac{n}{\ln 2}$$
$$= \frac{n}{\ln 2}$$

Therefore,

$$S(n) \leq \sum_{i=1}^{n} \log_2(\frac{n}{i}) + n$$
$$\leq \frac{n}{\ln 2} + n$$

$\Rightarrow S(n) = O(n)$