NP-Completeness

(Based on [Manber 1989])

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P vs. NP

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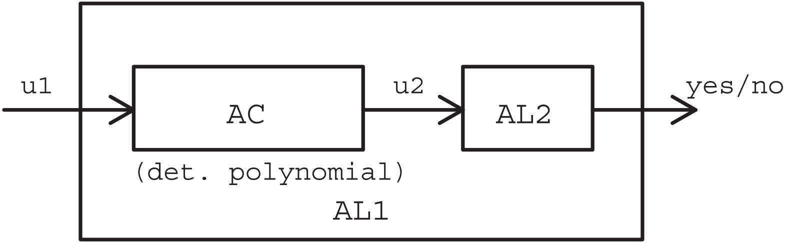
- P denotes the class of all problems that can be solved by *deterministic* algorithms in *polynomial* time.
- NP denotes the class of all problems that can be solved by *nondeterministic* algorithms in *polynomial* time.
- A nondeterministic algorithm, when faced with a choice of several options, has the power to guess the right one (if there is any).
- We will focus on *decision* problems, whose answer is either yes or no.

Decision as Language Recognition

- A *decision* problem can be viewed as a *language-recognition* problem.
- Let U be the set of all possible inputs to the decision problem and $L \subseteq U$ be the set of all inputs for which the answer to the problem is yes.
- We call L the *language* corresponding to the problem.
- The decision problem is to recognize whether a given input belongs to L.

Polynomial-Time Reductions

- Let L_1 and L_2 be two languages from the input spaces U_1 and U_2.
- We say that *L_1 is polynomially reducible to L_2* if there exists a conversion algorithm AC satisfying the following conditions:
  1. AC runs in polynomial time (deterministically).
  2. *$u_1 \in L_1$ if and only if $AC(u_1) = u_2 \in L_2$.*



Polynomial-Time Reductions (cont.)

Theorem (11.1)

If L_1 is polynomially reducible to L_2 and there is a polynomial-time algorithm for L_2, then there is a polynomial-time algorithm for L_1.

Theorem (11.2: transitivity)

If L_1 is polynomially reducible to L_2 and L_2 is polynomially reducible to L_3, then L_1 is polynomially reducible to L_3.

NP-Completeness

- A problem X is called an **NP-hard** problem if every problem in NP is polynomially reducible to X.
- A problem X is called an **NP-complete** problem if (1) X belongs to NP, and (2) X is NP-hard.

Lemma (11.3)

A problem X is an NP-complete problem if (1) X belongs to NP, and (2′) Y is polynomially reducible to X, for some NP-complete problem Y.

- If there exists an efficient (polynomial-time) algorithm for any NP-complete problem, then there exist efficient algorithms for all NP-complete (and hence all NP) problems.

The Satisfiability Problem (SAT)

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Problem

Given a Boolean expression in conjunctive normal form, determine whether it is satisfiable.

- A Boolean expression is in *conjunctive normal form* (CNF) if it is the product of several sums, e.g., $(x + y + \bar{z}) \cdot (\bar{x} + y + z) \cdot (\bar{x} + \bar{y} + \bar{z})$.
- A Boolean expression is said to be *satisfiable* if there exists an assignment of 0s and 1s to its variables such that the value of the expression is 1.

SAT (cont.)

Theorem (Cook's Theorem)

The SAT problem is NP-complete.

- This is our starting point for showing the NP-completeness of some other problems.
- Their NP-hardness will be proved by reduction directly or indirectly from SAT.

NP-Complete Problems

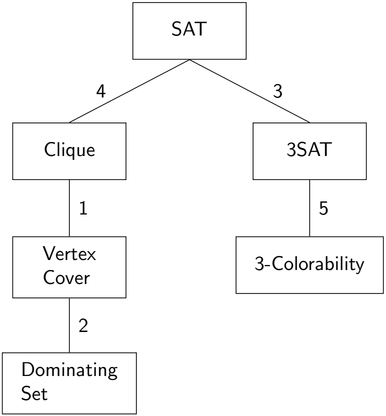

Figure: The order of NP-completeness proofs we will study.

Source: redrawn from [Manber 1989, Figure 11.1].

Vertex Cover

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Problem

Given an undirected graph $G = (V, E)$ and an integer k, determine whether G has a vertex cover containing $\leq k$ vertices.

A *vertex cover* of G is a set of vertices such that every edge in G is incident to at least one of these vertices.

Theorem (11.4)

The vertex-cover problem is NP-complete.

Main idea: by reduction from the clique problem.

Vertex Cover (cont.)

Proof outline:

- The vertex-cover problem is in NP, since given a graph we can guess a subset of vertices and check whether it contains $\leq k$ vertices and is indeed a vertex cover in ploynomial time.
- The clique problem, which is NP-complete, is polynomially reducible to the vertex-cover problem.
  - Let $G(V, E)$ and k represent an arbitrary instance of the clique problem.
  - Let $\overline{G}(V, \overline{E})$ be the complement of G; computing the complement of a graph takes only polynomial time.
  - Claim: G has a clique of size $\geq k$ iff $\overline{G}$ has a vertex cover of size $\leq |V| - k$.

Dominating Set

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Problem

Given an undirected graph $G = (V, E)$ and an integer k, determine whether G has a dominating set containing $\leq k$ vertices.

A *dominating set* D is a set of vertices such that every vertex of G is either in D or is adjacent to some vertex in D.

Theorem (11.5)

The dominating-set problem is NP-complete.

By reduction from the vertex-cover problem.

Dominating Set (cont.)

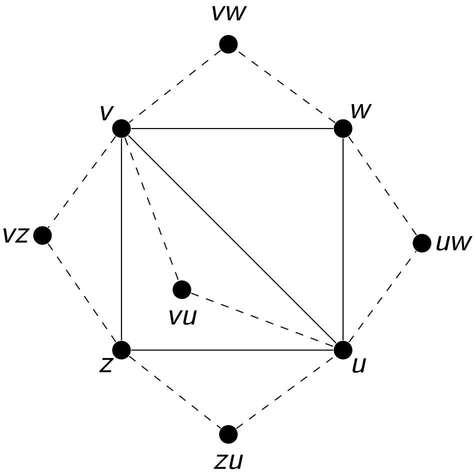

Figure: The dominating-set reduction.

Source: redrawn from [Manber 1989, Figure 11.2].

3SAT

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Problem

Given a Boolean expression in CNF such that each clause contains exactly three variables, determine whether it is satisfiable.

Theorem (11.6)

The 3SAT problem is NP-complete.

By reduction from the regular SAT problem.

3SAT (cont.)

- From an arbitrary clause $(x_1 + x_2 + \cdots + x_k)$, where $k \neq 3$, of the SAT problem to clauses of the 3SAT problem:
  - When $k \geq 4$,

$$
\begin{array}{r}
(x_1 + x_2 + y_1)\cdot \\
(x_3 + \overline{y_1} + y_2)\cdot \\
(x_4 + \overline{y_2} + y_3)\cdot \\
\vdots \\
(x_{k-2} + \overline{y_{k-4}} + y_{k-3})\cdot \\
(x_{k-1} + x_k + \overline{y_{k-3}})
\end{array}
$$

  - When $k = 2$,

$$(x_1 + x_2 + w) \cdot (x_1 + x_2 + \overline{w})$$

  - When $k = 1$,

$$(x_1 + y + z) \cdot (x_1 + \overline{y} + z) \cdot (x_1 + y + \overline{z}) \cdot (x_1 + \overline{y} + \overline{z})$$

Clique

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Problem

Given an undirected graph $G = (V, E)$ and an integer k, determine whether G contains a clique of size $\geq k$.

A *clique* C is a subgraph of G such that all vertices in C are adjacent to all other vertices in C.

Theorem (11.7)

The clique problem is NP-complete.

By reduction from the SAT problem.

Clique (cont.)

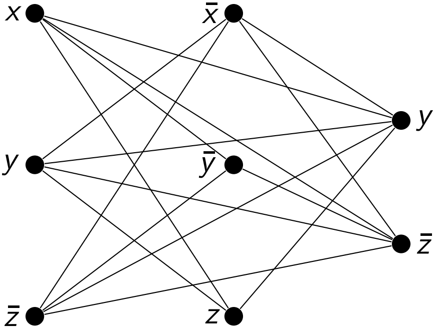

Figure: An example of the clique reduction for the expression $(x + y + \bar{z}) \cdot (\bar{x} + \bar{y} + z) \cdot (y + \bar{z})$.

Source: redrawn from [Manber 1989, Figure 11.3].

3-Coloring

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Problem

Given an undirected graph $G = (V, E)$, determine whether G can be colored with three colors.

Theorem (11.8)

The 3-coloring problem is NP-complete.

By reduction from the 3SAT problem.

3-Coloring (cont.)

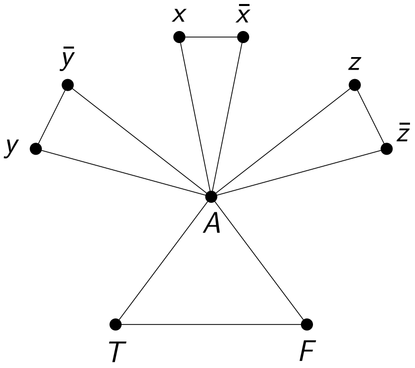

Figure: The first part of the construction in the reduction of 3SAT to 3-coloring.

Source: redrawn from [Manber 1989, Figure 11.4].

3-Coloring (cont.)

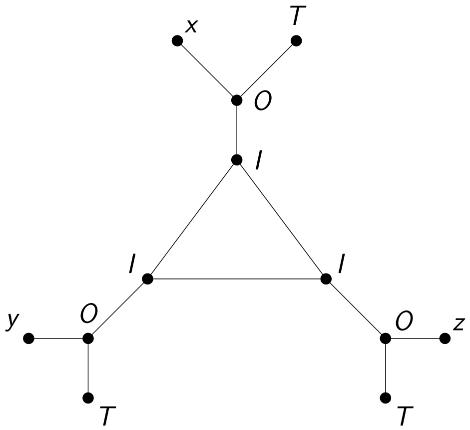

Figure: The subgraphs corresponding to the clauses in the reduction of 3SAT to 3-coloring.

Source: redrawn from [Manber 1989, Figure 11.5].

3-Coloring (cont.)

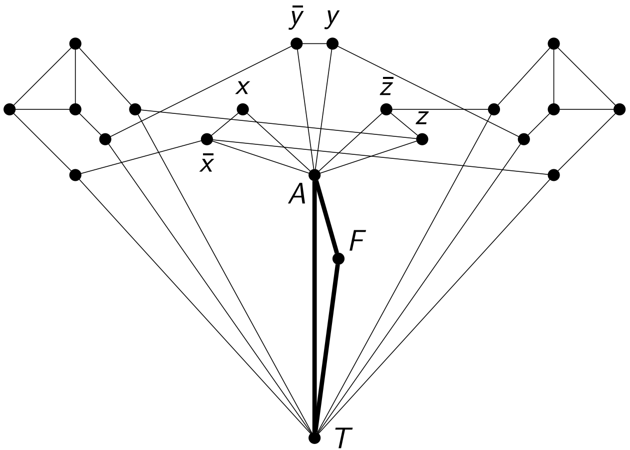

Figure: The graph corresponding to $(\bar{x} + \bar{y} + z) \cdot (\bar{x} + y + \bar{z})$.

Source: redrawn from [Manber 1989, Figure 11.6].

More NP-Complete Problems

- **Independent set**:
  An independent set in an undirected graph is a set of vertices no two of which are adjacent. The problem is to determine, given a graph G and an integer k, whether G contains an independent set with $\geq k$ vertices.
- **Hamiltonian cycle**:
  A Hamiltonian cycle in a graph is a (simple) cycle that contains each vertex exactly once. The problem is to determine whether a given graph contains a Hamiltonian cycle.
- **Travelling salesman**:
  The input includes a set of cities, the distances between all pairs of cities, and a number D. The problem is to determine whether there exists a (travelling-salesman) tour of all the cities having total length $\leq D$.

More NP-Complete Problems (cont.)

- **Partition**:
  The input is a set X where each element $x \in X$ has an associated size $s(x)$. The problem is to determine whether it is possible to partition the set into two subsets with exactly the same total size.
- **Knapsack**:
  The input is a set X, where each element $x \in X$ has an associated size $s(x)$ and value $v(x)$, and two other numbers S and V. The problem is to determine whether there is a subset $B \subseteq X$ whose total size is $\leq S$ and whose total value is $\geq V$.
- **Bin packing**:
  The input is a set of numbers $\{a_1, a_2, \cdots, a_n\}$ and two other numbers b and k. The problem is to determine whether the set can be partition into k subsets such that the sum of numbers in each subset is $\leq b$.