Design by Induction
(Based on [Manber 1989])

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Introduction

- It is not necessary to design the steps required to solve a problem from scratch.
- It is sufficient to guarantee the following:
  1. It is possible to solve one small instance or a few small instances of the problem. (base case)
  2. A solution to every problem/instance can be constructed from solutions to smaller problems/instances. (inductive step)

Evaluating Polynomials

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Problem

Given a sequence of real numbers $a_n, a_{n-1}, \cdots, a_1, a_0$, and a real number x, compute the value of the polynomial

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

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Motivation: different approaches to the inductive step may result in algorithms of very different time complexities.

Evaluating Polynomials (cont.)

- Let $P_{n-1}(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.
- **Induction hypothesis** (first attempt)
  We know how to evaluate a polynomial represented by the input $a_{n-1}, \cdots, a_1, a_0$, at the point x, i.e., we know how to compute $P_{n-1}(x)$.
- $P_n(x) = a_n x^n + P_{n-1}(x)$.

Evaluating Polynomials (cont.)

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  We know how to evaluate a polynomial represented by the input $a_{n-1}, \cdots, a_1, a_0$, at the point x, i.e., we know how to compute $P_{n-1}(x)$.
- $P_n(x) = a_n x^n + P_{n-1}(x)$.
- Number of multiplications:

$$n + (n-1) + \cdots + 2 + 1 = \frac{n(n+1)}{2}.$$

Evaluating Polynomials (cont.)

- **Induction hypothesis** (second attempt)
  We know how to compute $P_{n-1}(x)$, and we know how to compute x^{n-1}.
- $P_n(x) = a_n x(x^{n-1}) + P_{n-1}(x)$.

Evaluating Polynomials (cont.)

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- **Induction hypothesis** (second attempt)
  We know how to compute $P_{n-1}(x)$, and we know how to compute x^{n-1}.
- $P_n(x) = a_n x(x^{n-1}) + P_{n-1}(x)$.
- Number of multiplications: $2(n-1) + 1 = 2n - 1$.

$$\overbrace{a_n x(x^{n-1}) + a_{n-1}x(x^{n-2}) + \cdots + a_2 x(x^1)}^{2(n-1) \text{ multiplications}} + \overbrace{a_1 x + a_0}^{1 \text{ multiplication}}$$

Evaluating Polynomials (cont.)

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- Let $P'_{n-1}(x) = a_n x^{n-1} + a_{n-1} x^{n-2} + \cdots + a_1$.
- **Induction hypothesis** (final attempt)
  We know how to evaluate a polynomial represented by the coefficients $a_n, a_{n-1}, \cdots, a_1$, at the point x, i.e., we know how to compute $P'_{n-1}(x)$.
- $P_n(x) = P'_n(x) = P'_{n-1}(x) \cdot x + a_0$.

Evaluating Polynomials (cont.)

- More generally,

$$\begin{cases} P'_0(x) = a_n \\ P'_i(x) = P'_{i-1}(x) \cdot x + a_{n-i}, \text{ for } 1 \leq i \leq n \end{cases}$$

Evaluating Polynomials (cont.)

- More generally,

$$\begin{cases} P'_0(x) = a_n \\ P'_i(x) = P'_{i-1}(x) \cdot x + a_{n-i}, \ \text{for } 1 \leq i \leq n \end{cases}$$

- Number of multiplications: n.

Evaluating Polynomials (cont.)

Algorithm Polynomial_Evaluation $(\bar{a}, x)$;
begin
    $P := a_n$;
    for $i := 1$ **to** n **do**
        $P := x * P + a_{n-i}$
end

This algorithm is known as *Horner's rule*.

Maximal Induced Subgraph

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Problem

Given an undirected graph $G = (V, E)$ and an integer k, find an induced subgraph $H = (U, F)$ of G of maximum size such that all vertices of H have degree $\geq k$ (in H), or conclude that no such induced subgraph exists.

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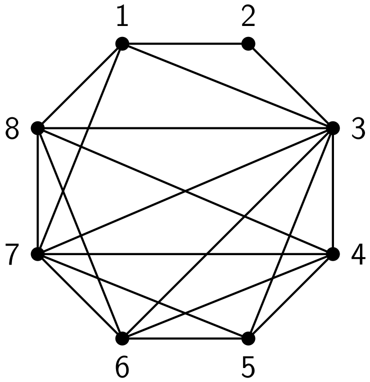
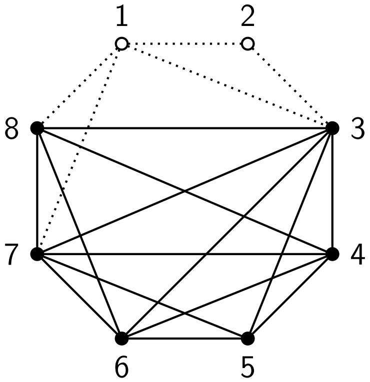
Design Idea: in the inductive step, we try to remove one vertex (that cannot possibly be part of the solution) to get a smaller instance.

Maximal Induced Subgraph (cont.)



A graph G of eight nodes.



Maximal induced subgraph of G when $k = 4$.

Maximal Induced Subgraph (cont.)

- Recursive:

Algorithm Max_Ind_Subgraph (G, k);
begin
  if the degree of every vertex of $G \geq k$ **then**
    Max_Ind_Subgraph := G;
  else let v be a vertex of G with degree $< k$;
      Max_Ind_Subgraph := Max_Ind_Subgraph($G - v, k$);
end

Maximal Induced Subgraph (cont.)

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  end

- Iterative:

  Algorithm Max_Ind_Subgraph (G, k);
  begin
      while the degree of some vertex v of $G < k$ **do**
          $G := G - v$;
      Max_Ind_Subgraph := G;
  end

One-to-One Mapping

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Problem

Given a finite set A and a mapping f from A to itself, find a subset $S \subseteq A$ with the maximum number of elements, such that
(1) the function f maps every element of S to another element of S (i.e., f maps S into itself), and
(2) no two elements of S are mapped to the same element (i.e., f is one-to-one when restricted to S).

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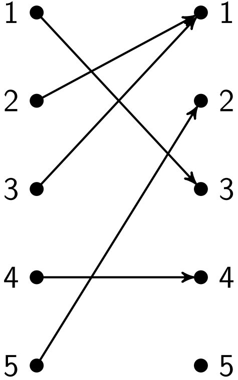
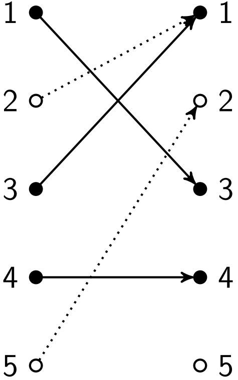
An element that is not mapped to may be removed.

One-to-One Mapping (cont.)



A given set A and a mapping to itself.



The maximal selected subset S and the remaining 1-to-1 mapping.

One-to-One Mapping (cont.)

Algorithm Mapping (f, n);
begin
    $S := A$;
    for $j := 1$ **to** n **do** $c[j] := 0$;
    for $j := 1$ **to** n **do** increment $c[f[j]]$;
    for $j := 1$ **to** n **do**
        if $c[j] = 0$ **then** put j in Queue;
    while Queue not empty **do**
        remove i from the top of Queue;
        $S := S - \{i\}$;
        decrement $c[f[i]]$;
        if $c[f[i]] = 0$ **then** put $f[i]$ in Queue
end

Celebrity

Problem

Given an $n \times n$ adjacency matrix, determine whether there exists an i (the "celebrity") such that all the entries in the i-th column (except for the ii-th entry) are 1*, and all the entries in the i-th row (except for the ii-th entry) are* 0*.*

Note: A celebrity corresponds to a sink of the directed graph.

Note: Every directed graph has at most one sink.

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Motivation: the trivial solution has a time complexity of $O(n^2)$. Can we do better, in $O(n)$?

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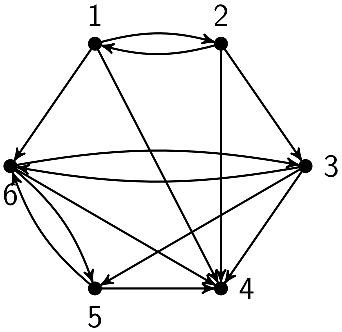
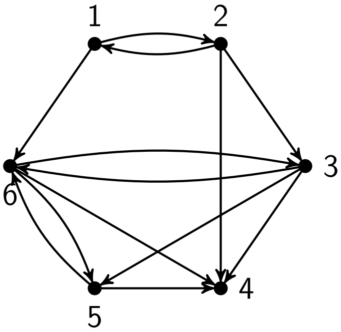
To achieve $O(n)$ time, we must reduce the problem size by at least one in constant time.

Celebrity (cont.)



A graph of six nodes
with a sink (node 4).



A graph of six nodes
without a sink.

Celebrity (cont.)

Basic idea: check whether i knows j.

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The $O(n)$ algorithm proceeds in two stages:

- Eliminate a node every round until only one is left.
- Check whether the remaining one is truly a celebrity.

Celebrity (cont.)

```
Algorithm Celebrity (Know);
begin
    i := 1;
    j := 2;
    next := 3;
    while next ≤ n + 1 do
        if Know[i, j] then i := next
                      else j := next;
        next := next + 1;
    if i = n + 1 then candidate := j
                 else candidate := i;
```

Celebrity (cont.)

$wrong := false;$
$k := 1;$
$Know[candidate, candidate] := false;$
while $not\ wrong\ and\ k \le n$ **do**
    if $Know[candidate, k]$ **then** $wrong := true;$
    if $not\ Know[k, candidate]$ **then**
        if $candidate \neq k$ **then** $wrong := true;$
    $k := k + 1;$
if $not\ wrong$ **then** $celebrity := candidate$
                else $celebrity := 0;$
end

The Skyline Problem

Problem

Given the exact locations and shapes of several rectangular buildings in a city, draw the skyline (in two dimension) of these buildings, eliminating hidden lines.

Motivation: different approaches to the inductive step may result in algorithms of very different time complexities.

The Skyline Problem

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Given the exact locations and shapes of several rectangular buildings in a city, draw the skyline (in two dimension) of these buildings, eliminating hidden lines.

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Compare: adding buildings one by one to an existing skyline vs. merging two skylines of about the same size

The Skyline Problem

- Adding one building at a time:

$$\begin{cases} T(1) = O(1) \\ T(n) = T(n-1) + O(n), n \geq 2 \end{cases}$$

  Time complexity: $O(n^2)$.

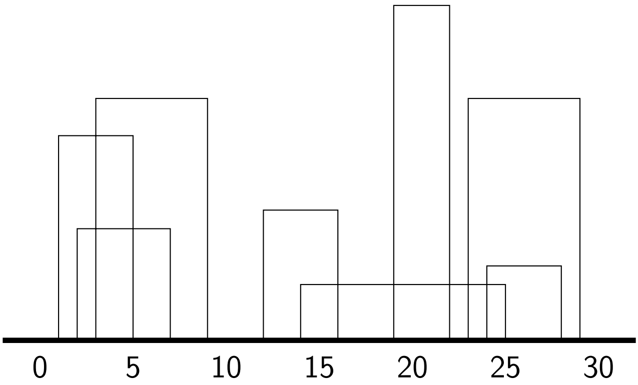
- Merging two skylines every round:

$$\begin{cases} T(1) = O(1) \\ T(n) = 2T(\frac{n}{2}) + O(n), n \geq 2 \end{cases}$$

  Time complexity: $O(n \log n)$.

Representation of a Skyline

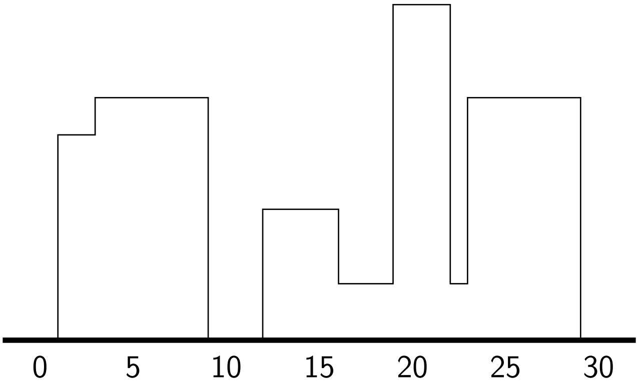
Input: (1,**11**,5), (2,**6**,7), (3,**13**,9), (12,**7**,16), (14,**3**,25), (19,**18**,22), (23,**13**,29), and (24,**4**,28).



Source: adapted from [Manber 1989, Figure 5.5(a)].

Representation of a Skyline (cont.)

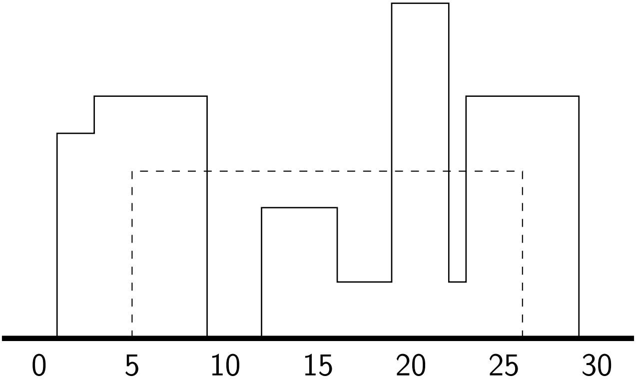
Representation: (1,**11**,3,**13**,9,**0**,12,**7**,16,**3**,19,**18**,22,**3**,23,**13**,29).



Source: adapted from [Manber 1989, Figure 5.5(b)].

Adding a Building

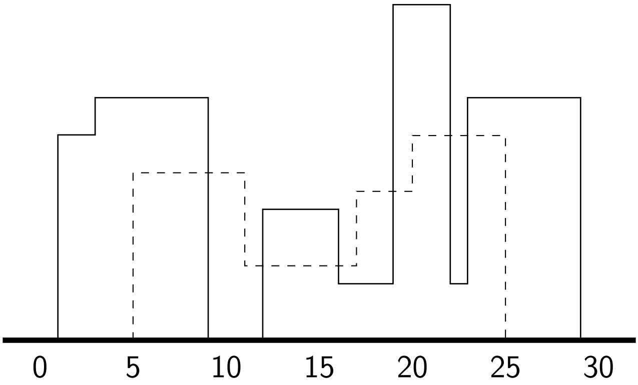
- Add (5,**9**,26) to (1,**11**,3,**13**,9,**0**,12,**7**,16,**3**,19,**18**,22,**3**,23,**13**,29).



Source: adapted from [Manber 1989, Figure 5.6].

- The skyline becomes (1,**11**,3,**13**,9,**9**,19,**18**,22,**9**,23,**13**,29).

Merging Two Skylines



Source: adapted from [Manber 1989, Figure 5.7].

Balance Factors in Binary Trees

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Problem

Given a binary tree T with n nodes, compute the balance factors of all nodes.

The balance factor of a node is defined as the difference between the height of the node's left subtree and the height of the node's right subtree.

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Problem

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Motivation: an example of why we must strengthen the hypothesis (and hence the problem to be solved).

Balance Factors in Binary Trees (cont.)

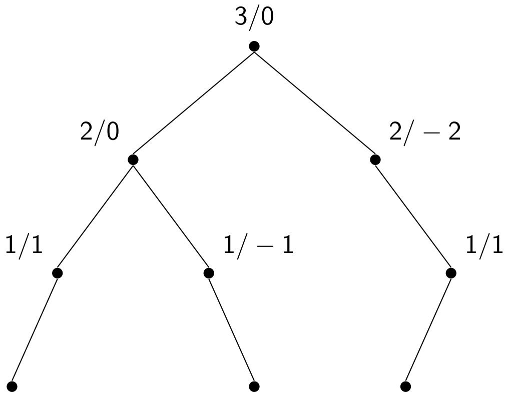

Figure: A binary tree. The numbers represent h/b, where h is the height and b is the balance factor.

Source: redrawn from [Manber 1989, Figure 5.8].

Balance Factors in Binary Trees (cont.)

- **Induction hypothesis**
  We know how to compute balance factors of all nodes in trees that have $< n$ nodes.

Balance Factors in Binary Trees (cont.)

- **Induction hypothesis**
  We know how to compute balance factors of all nodes in trees that have $< n$ nodes.
- **Stronger induction hypothesis**
  We know how to compute balance factors and heights of all nodes in trees that have $< n$ nodes.

Maximum Consecutive Subsequence

Problem

Given a sequence $x_1, x_2, \cdots, x_n$ *of real numbers (not necessarily positive), find a subsequence* $x_i, x_{i+1}, \cdots, x_j$ *(of consecutive elements) such that the sum of the numbers in it is maximum over all subsequences of consecutive elements.*

Example:
In the sequence $(2, -3, 1.5, -1, 3, -2, -3, 3)$, the maximum subsequence is $(1.5, -1, 3)$.

Maximum Consecutive Subsequence

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Problem

Given a sequence $x_1, x_2, \cdots, x_n$ of real numbers (not necessarily positive), find a subsequence $x_i, x_{i+1}, \cdots, x_j$ (of consecutive elements) such that the sum of the numbers in it is maximum over all subsequences of consecutive elements.

Example:
In the sequence $(2, -3, 1.5, -1, 3, -2, -3, 3)$, the maximum subsequence is $(1.5, -1, 3)$.

Motivation: another example of strengthening the hypothesis.

Maximum Consecutive Subsequence (cont.)

- **Induction hypothesis**
  We know how to find the maximum subsequence in sequences of size $< n$.

Maximum Consecutive Subsequence (cont.)

- **Induction hypothesis**
  We know how to find the maximum subsequence in sequences of size $< n$.
- **Stronger induction hypothesis**
  We know how to find, in sequences of size $< n$, the maximum subsequence overall and the maximum subsequence that is a suffix.

Maximum Consecutive Subsequence (cont.)

- **Induction hypothesis**
  We know how to find the maximum subsequence in sequences of size $< n$.
- **Stronger induction hypothesis**
  We know how to find, in sequences of size $< n$, the maximum subsequence overall and the maximum subsequence that is a suffix.

  Reasoning: the maximum subsequence of problem size n is obtained either
  - directly from the maximum subsequence of problem size $n - 1$ or
  - from appending the n-th element to the maximum suffix of problem size $n - 1$.

Maximum Consecutive Subsequence (cont.)

```
Algorithm Max_Consec_Subseq (X, n);
begin
    Global_Max := 0;
    Suffix_Max := 0;
    for i := 1 to n do
        if x[i] + Suffix_Max > Global_Max then
            Suffix_Max := Suffix_Max + x[i];
            Global_Max := Suffix_Max
        else if x[i] + Suffix_Max > 0 then
                Suffix_Max := Suffix_Max + x[i]
        else Suffix_Max := 0
end
```

The Knapsack Problem

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Problem

Given an integer K and n items of different sizes such that the i-th item has an integer size k_i, find a subset of the items whose sizes sum to exactly K, or determine that no such subset exists.

The Knapsack Problem

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Given an integer K and n items of different sizes such that the i-th item has an integer size k_i, find a subset of the items whose sizes sum to exactly K, or determine that no such subset exists.

Design Idea: use strong induction so that solutions to all smaller instances may be used.

The Knapsack Problem (cont.)

- Let $P(n, K)$ denote the problem where n is the number of items and K is the size of the knapsack.
- **Induction hypothesis**
  We know how to solve $P(n-1, K)$.

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- **Induction hypothesis**
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- **Stronger induction hypothesis**
  We know how to solve $P(n-1, k)$, for all $0 \leq k \leq K$.

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- **Induction hypothesis**
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  We know how to solve $P(n-1, k)$, for all $0 \leq k \leq K$.

  Reasoning: $P(n, K)$ has a solution if either
  - $P(n-1, K)$ has a solution or
  - $P(n-1, K-k_n)$ does, provided $K-k_n \geq 0$.

The Knapsack Problem (cont.)

An example of the table constructed for the knapsack problem:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	O	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$k_1 = 2$	O	-	I	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$k_2 = 3$	O	-	O	I	-	I	-	-	-	-	-	-	-	-	-	-	-
$k_3 = 5$	O	-	O	O	-	O	-	I	I	-	I	-	-	-	-	-	-
$k_4 = 6$	O	-	O	O	-	O	I	O	O	I	O	I	-	I	I	-	I

"I": a solution containing this item has been found.
"O": a solution without this item has been found.
"-": no solution has yet been found.

Source: adapted from [Manber 1989, Figure 5.11].

The Knapsack Problem (cont.)

```
Algorithm Knapsack (S, K);
    P[0, 0].exist := true;
    for k := 1 to K do
        P[0, k].exist := false;
    for i := 1 to n do
        for k := 0 to K do
            P[i, k].exist := false;
            if P[i − 1, k].exist then
                P[i, k].exist := true;
                P[i, k].belong := false
            else if k − S[i] ≥ 0 then
                    if P[i − 1, k − S[i]].exist then
                        P[i, k].exist := true;
                        P[i, k].belong := true
```