Searching and Sorting

(Based on [Manber 1989])

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Searching a Sorted Sequence

Problem

Let $x_1, x_2, \cdots, x_n$ *be a sequence of real numbers such that* $x_1 \leq x_2 \leq \cdots \leq x_n$. *Given a real number* z, *we want to find whether* z *appears in the sequence, and, if it does, to find an index* i *such that* $x_i = z$.

Searching a Sorted Sequence

Problem

Let $x_1, x_2, \cdots, x_n$ be a sequence of real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$. Given a real number z, we want to find whether z appears in the sequence, and, if it does, to find an index i such that $x_i = z$.

Idea: cut the search space in half by asking only one question.

$$\begin{cases} T(1) = O(1) \\ T(n) = T(\frac{n}{2}) + O(1), n \geq 2 \end{cases}$$

Time complexity: $O(\log n)$ (applying the master theorem with $a = 1$, $b = 2$, $k = 0$, and $b^k = 1 = a$).

Binary Search

```
function Find (z, Left, Right) : integer;
begin
    if Left = Right then
        if X[Left] = z then Find := Left
        else Find := 0
    else
        Middle := ⌈(Left+Right)/2⌉;
        if z < X[Middle] then
            Find := Find(z, Left, Middle − 1)
        else
            Find := Find(z, Middle, Right)
end

Algorithm Binary_Search (X, n, z);
begin
    Position := Find(z, 1, n);
end
```

Binary Search: Alternative

function Find $(z, Left, Right) : integer;$
begin
    if $Left > Right$ **then**
        $Find := 0$
    else
        $Middle := \lceil \frac{Left+Right}{2} \rceil;$
        if $z = X[Middle]$ **then**
            $Find := Middle$
        else if $z < X[Middle]$ **then**
            $Find := Find(z, Left, Middle - 1)$
        else
            $Find := Find(z, Middle + 1, Right)$
end

How do the two algorithms compare?

Searching a Cyclically Sorted Sequence

Problem

Given a cyclically sorted list, find the position of the minimal element in the list (we assume, for simplicity, that this position is unique).

- Example 1:
1	2	3	4	5	6	7	8
5	6	7	0	1	2	3	4
  - The 4th is the minimal element.
- Example 2:
1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7
  - The 1st is the minimal element.

Searching a Cyclically Sorted Sequence

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- Example 2:
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0	1	2	3	4	5	6	7
  - The 1st is the minimal element.
- To cut the search space in half, what question should we ask?

Cyclic Binary Search

Algorithm Cyclic_Binary_Search (X, n);
begin
    $Position := Cyclic\_Find(1, n)$;
end

function Cyclic_Find $(Left, Right)$: $integer$;
begin
    if $Left = Right$ **then** $Cyclic\_Find := Left$
    else
        $Middle := \lfloor \frac{Left+Right}{2} \rfloor$;
        if $X[Middle] < X[Right]$ **then**
            $Cyclic\_Find := Cyclic\_Find(Left, Middle)$
        else
            $Cyclic\_Find := Cyclic\_Find(Middle + 1, Right)$
end

“Fixpoints”

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Problem

Given a sorted sequence of distinct integers $a_1, a_2, \cdots, a_n$, *determine whether there exists an index* i *such that* $a_i = i$.

- Example 1:
  - $$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& -1 & 1 & 2 & 4 & 5 & 6 & 8 & 9 &] \end{array}$$
  - $a_4 = 4$ (there are more ...).
- Example 2:
  - $$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& -1 & 1 & 2 & 5 & 6 & 8 & 9 & 10 &] \end{array}$$
  - There is no i such that $a_i = i$.

"Fixpoints"

Problem

Given a sorted sequence of distinct integers $a_1, a_2, \cdots, a_n$*, determine whether there exists an index* i *such that* $a_i = i$*.*

- Example 1:
  - $$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& -1 & 1 & 2 & 4 & 5 & 6 & 8 & 9 &] \end{array}$$
  - $a_4 = 4$ (there are more ...).
- Example 2:
  - $$\begin{array}{cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ [& -1 & 1 & 2 & 5 & 6 & 8 & 9 & 10 &] \end{array}$$
  - There is no i such that $a_i = i$.
- Again, can we cut the search space in half by asking only one question?

A Special Binary Search

function Special_Find $(Left, Right)$: *integer*;
begin
    if $Left = Right$ **then**
        if $A[Left] = Left$ **then** $Special\_Find := Left$
        else $Special\_Find := 0$
    else
        $Middle := \lfloor \frac{Left+Right}{2} \rfloor$;
        if $A[Middle] < Middle$ **then**
            $Special\_Find := Special\_Find(Middle + 1, Right)$
        else
            $Special\_Find := Special\_Find(Left, Middle)$
end

A Special Binary Search (cont.)

Algorithm Special_Binary_Search (A, n);
begin
    $Position := Special\_Find(1, n)$;
end

Stuttering Subsequence

Problem

Given two sequences A $(= a_1 a_2 \cdots a_n)$ and B $(= b_1 b_2 \cdots b_m)$, find the maximal value of i such that B^i is a subsequence of A.

- If $B = xyzzx$, then $B^2 = xxyyzzzzxx$, $B^3 = xxxyyyzzzzzzxxx$, etc.
- B is a subsequence of A if we can embed B inside A in the same order but with possible holes.
- For example, $B^2 = xxyyzzzzxx$ is a subsequence of $xxzzyyyyxxzzzzzxxx$.

Stuttering Subsequence

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- If B^j is a subsequence of A, then B^i is a subsequence of A, for $1 \leq i \leq j$.

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- B is a subsequence of A if we can embed B inside A in the same order but with possible holes.
- For example, $B^2 = xxyyzzzzxx$ is a subsequence of $xxzzyyyyxxzzzzzxxx$.
- If B^j is a subsequence of A, then B^i is a subsequence of A, for $1 \leq i \leq j$.
- The maximum value of i cannot exceed $\lfloor \frac{n}{m} \rfloor$ (or B^i would be longer than A).

Stuttering Subsequence (cont.)

Two ways to find the maximum i:

- Sequential search: try 1, 2, 3, etc. sequentially.

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  Time complexity: $O(nj)$, where j is the maximum value of i.
- Binary search between 1 and $\lfloor \frac{n}{m} \rfloor$.

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Can binary search be applied, if the bound $\lfloor \frac{n}{m} \rfloor$ is unknown?

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Think of the base case in a reversed induction.

Interpolation Search

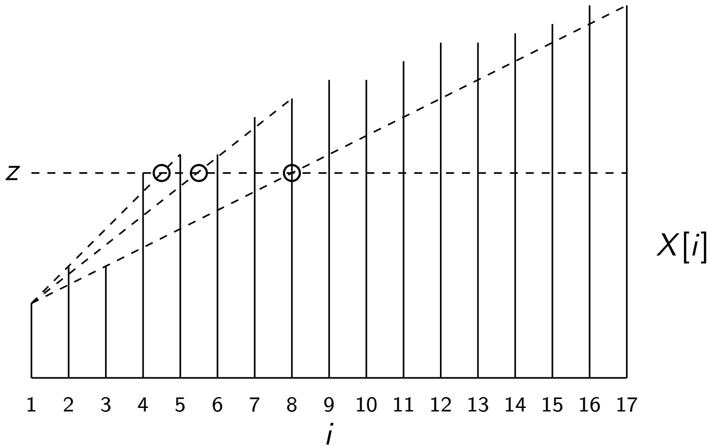
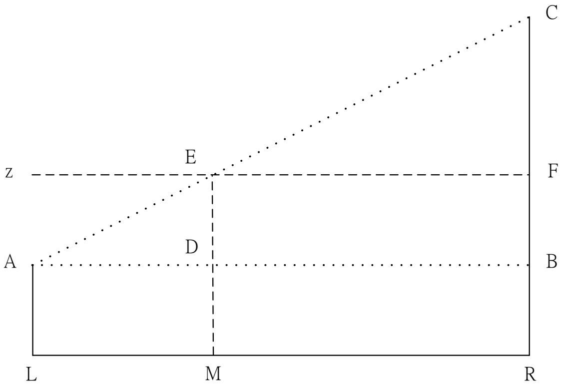


Figure: Interpolation search.

Source: redrawn from [Manber 1989, Figure 6.4].

Interpolation Search (cont.)



$$\frac{\overline{LM}}{\overline{LR}} = \frac{\overline{AD}}{\overline{AB}} = \frac{\overline{AE}}{\overline{AC}} = \frac{\overline{BF}}{\overline{BC}}, \text{so } |\overline{LM}| = \frac{|\overline{BF}|}{|\overline{BC}|} \times |\overline{LR}|$$

Interpolation Search (cont.)

function Int_Find $(z, Left, Right)$: *integer*;
begin
    if $X[Left] = z$ **then** $Int\_Find := Left$
    else if $Left = Right$ or $X[Left] = X[Right]$ **then**
        $Int\_Find := 0$
    else
        $Next\_Guess := \lceil Left + \frac{(z - X[Left])(Right - Left)}{X[Right] - X[Left]} \rceil$;
        if $z < X[Next\_Guess]$ **then**
            $Int\_Find := Int\_Find(z, Left, Next\_Guess - 1)$
        else
            $Int\_Find := Int\_Find(z, Next\_Guess, Right)$
end

Interpolation Search (cont.)

Algorithm Interpolation_Search (X, n, z);
begin
  if $z < X[1]$ or $z > X[n]$ **then** $Position := 0$
  else $Position := Int\_Find(z, 1, n)$;
end

Sorting

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Problem

Given n numbers $x_1, x_2, \cdots, x_n$, arrange them in increasing order. In other words, find a sequence of distinct indices $1 \leq i_1, i_2, \cdots, i_n \leq n$, such that $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_n}$.

A sorting algorithm is called **in-place** if no additional work space is used besides the initial array that holds the elements.

Using Balanced Search Trees

- Balanced search trees, such as AVL trees, may be used for sorting:
    1. Create an empty tree.
    2. Insert the numbers one by one to the tree.
    3. Traverse the tree and output the numbers.

Using Balanced Search Trees

- Balanced search trees, such as AVL trees, may be used for sorting:
  1. Create an empty tree.
  2. Insert the numbers one by one to the tree.
  3. Traverse the tree and output the numbers.
- What's the time complexity? Suppose we use an AVL tree.

Radix Sort

```
Algorithm Straight_Radix (X, n, k);
begin
    put all elements of X in a queue GQ;
    for i := 1 to d do /* A digit may be 1, 2, ..., or d. */
        initialize queue Q[i] to be empty
    for i := k downto 1 do
        while GQ is not empty do
            pop x from GQ;
            c := the i-th digit of x;
            insert x into Q[c];
        for t := 1 to d do
            insert Q[t] into GQ;
    for i := 1 to n do
        pop X[i] from GQ
end
```

Radix Sort

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    for i := k downto 1 do
        while GQ is not empty do
            pop x from GQ;
            c := the i-th digit of x;
            insert x into Q[c];
        for t := 1 to d do
            insert Q[t] into GQ;
    for i := 1 to n do
        pop X[i] from GQ
end
```

Time complexity: $O(nk)$.

Merge Sort

Algorithm Mergesort (X, n);
begin $M\_Sort(1, n)$ **end**

procedure M_Sort $(Left, Right)$;
begin
  if $Right - Left = 1$ **then**
    if $X[Left] > X[Right]$ **then** $swap(X[Left], X[Right])$
  else if $Left \neq Right$ **then**
    $Middle := \lceil \frac{1}{2}(Left + Right) \rceil$;
    $M\_Sort(Left, Middle - 1)$;
    $M\_Sort(Middle, Right)$;

Merge Sort (cont.)

$i := Left;\ \ j := Middle;\ \ k := 0;$
while $(i \leq Middle - 1)$ and $(j \leq Right)$ **do**
  $k := k + 1;$
  if $X[i] \leq X[j]$ **then**
    $TEMP[k] := X[i];\ \ i := i + 1$
  else $TEMP[k] := X[j];\ \ j := j + 1;$
if $j > Right$ **then**
  for $t := 0$ **to** $Middle - 1 - i$ **do**
    $X[Right - t] := X[Middle - 1 - t]$
for $t := 0$ **to** $k - 1$ **do**
  $X[Left + t] := TEMP[1 + t]$
end

Merge Sort (cont.)

```
i := Left;  j := Middle;  k := 0;
while (i ≤ Middle − 1) and (j ≤ Right) do
    k := k + 1;
    if X[i] ≤ X[j] then
        TEMP[k] := X[i];  i := i + 1
    else TEMP[k] := X[j];  j := j + 1;
if j > Right then
    for t := 0 to Middle − 1 − i do
        X[Right − t] := X[Middle − 1 − t]
for t := 0 to k − 1 do
    X[Left + t] := TEMP[1 + t]
end
```

Time complexity: $O(n \log n)$.

Merge Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
②	⑥	8	5	10	9	12	1	15	7	3	13	4	11	16	14
2	6	⑤	⑧	10	9	12	1	15	7	3	13	4	11	16	14
②	⑤	⑥	⑧	10	9	12	1	15	7	3	13	4	11	16	14
2	5	6	8	⑨	⑩	12	1	15	7	3	13	4	11	16	14
2	5	6	8	9	10	①	⑫	15	7	3	13	4	11	16	14
2	5	6	8	①	⑨	⑩	⑫	15	7	3	13	4	11	16	14
①	②	⑤	⑥	⑧	⑨	⑩	⑫	15	7	3	13	4	11	16	14
1	2	5	6	8	9	10	12	⑦	⑮	3	13	4	11	16	14
1	2	5	6	8	9	10	12	7	15	③	⑬	4	11	16	14
1	2	5	6	8	9	10	12	③	⑦	⑬	⑮	4	11	16	14
1	2	5	6	8	9	10	12	3	7	13	15	④	⑪	16	14
1	2	5	6	8	9	10	12	3	7	13	15	4	11	⑭	⑯
1	2	5	6	8	9	10	12	3	7	13	15	④	⑪	⑭	⑯
1	2	5	6	8	9	10	12	③	④	⑦	⑪	⑬	⑭	⑮	⑯
①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫	⑬	⑭	⑮	⑯

Figure: An example of mergesort.

Source: redrawn from [Manber 1989, Figure 6.8].

Quick Sort

Algorithm Quicksort (X, n);
begin
    $Q\_Sort(1, n)$
end

procedure Q_Sort $(Left, Right)$;
begin
    if $Left < Right$ **then**
        $Partition(X, Left, Right)$;
        $Q\_Sort(Left, Middle - 1)$;
        $Q\_Sort(Middle + 1, Right)$
end

Quick Sort

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    if $Left < Right$ **then**
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        $Q\_Sort(Left, Middle - 1)$;
        $Q\_Sort(Middle + 1, Right)$
end

Time complexity: $O(n^2)$, but $O(n \log n)$ in average

Quick Sort (cont.)

Algorithm Partition$(X, Left, Right)$;
begin
    $pivot := X[Left]$;
    $L := Left + 1$; $R := Right$;
    while $L \leq R$ **do**
        while $L \leq Right$ and $X[L] \leq pivot$ **do** $L := L + 1$;
        while $R \geq Left$ and $X[R] > pivot$ **do** $R := R - 1$;
        if $L < R$ **then**
            $swap(X[L], X[R])$;
            $L := L + 1$;
            $R := R - 1$;
    $Middle := R$;
    $swap(X[Left], X[Middle])$
end

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
6	2	(4)	5	10	9	12	1	15	7	3	13	(8)	11	16	14
6	2	4	5	(3)	9	12	1	15	7	(10)	13	8	11	16	14
6	2	4	5	3	(1)	12	(9)	15	7	10	13	8	11	16	14
(1)	2	4	5	3	(6)	12	9	15	7	10	13	8	11	16	14

Figure: Partition of an array around the pivot 6.

Source: redrawn from [Manber 1989, Figure 6.10].

Quick Sort (cont.)

6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
1	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	2	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	②	4	5	3	⑥	12	9	15	7	10	13	8	11	16	14
①	②	3	④	5	⑥	12	9	15	7	10	13	8	11	16	14
①	②	3	④	5	⑥	8	9	11	7	10	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	11	9	10	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	10	9	⑪	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	13	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	⑬	15	16	14
①	②	3	④	5	⑥	7	⑧	9	⑩	⑪	⑫	⑬	14	⑮	16

Figure: An example of quicksort.

Source: redrawn from [Manber 1989, Figure 6.12].

Average-Case Complexity of Quick Sort

- When $X[i]$ is selected (at random) as the pivot,

$$T(n) = n + 1 + T(i-1) + T(n-i), \text{ where } n \geq 2.$$

($T(0) = 0$ and $T(1) = 0$.)

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- When $X[i]$ is selected (at random) as the pivot,

$$T(n) = n + 1 + T(i-1) + T(n-i), \text{ where } n \geq 2.$$

($T(0) = 0$ and $T(1) = 0$.)

The average running time will then be

$$\begin{aligned} T(n) &= n + 1 + \tfrac{1}{n} \textstyle\sum_{i=1}^{n} (T(i-1) + T(n-i)) \\ &= n + 1 + \tfrac{1}{n} \textstyle\sum_{i=1}^{n} T(i-1) + \tfrac{1}{n} \textstyle\sum_{i=1}^{n} T(n-i) \\ &= n + 1 + \tfrac{1}{n} \textstyle\sum_{j=0}^{n-1} T(j) + \tfrac{1}{n} \textstyle\sum_{j=0}^{n-1} T(j) \\ &= n + 1 + \tfrac{2}{n} \textstyle\sum_{i=0}^{n-1} T(i) \\ &= n + 1 + \tfrac{2}{n} \textstyle\sum_{i=1}^{n-1} T(i) \end{aligned}$$

- Solving this recurrence relation with full history, $T(n) = O(n \log n)$.

Heap Sort

Algorithm Heapsort (A, n);
begin
    $\mathit{Build\_Heap}(A)$;
    for $i := n$ **downto** 2 **do**
        $\mathit{swap}(A[1], A[i])$;
        $\mathit{Rearrange\_Heap}(i-1)$
end

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    for $i := n$ **downto** 2 **do**
        $swap(A[1], A[i])$;
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end

Time complexity: $O(n \log n)$

Heap Sort (cont.)

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procedure Rearrange_Heap (k);
begin
    $parent := 1$;
    $child := 2$;
    while $child \leq k$ **do**
        if $child + 1 \leq k$ and $A[child] < A[child + 1]$ **then**
            $child := child + 1$;
        if $A[child] > A[parent]$ **then**
            $swap(A[parent], A[child])$;
            $parent := child$;
            $child := 2 * child$
        else $child := k + 1$
end

Heap Sort (cont.)

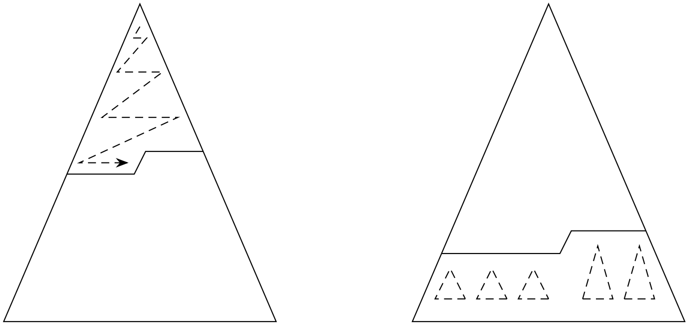

Figure: Top down and bottom up heap construction.

Source: redrawn from [Manber 1989, Figure 6.14].

Heap Sort (cont.)

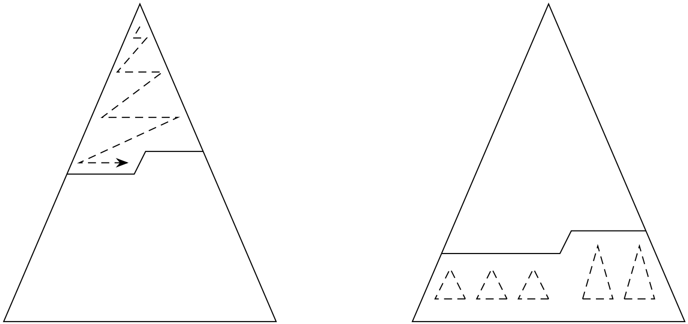

Figure: Top down and bottom up heap construction.

Source: redrawn from [Manber 1989, Figure 6.14].

How do the two approaches compare?

Building a Heap Bottom Up

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
6	2	8	5	10	9	12	1	15	7	3	13	4	11	16	14
6	2	8	5	10	9	12	14	15	7	3	13	4	11	16	1
6	2	8	5	10	9	16	14	15	7	3	13	4	11	12	1
6	2	8	5	10	13	16	14	15	7	3	9	4	11	12	1
6	2	8	5	10	13	16	14	15	7	3	9	4	11	12	1
6	2	8	15	10	13	16	14	5	7	3	9	4	11	12	1
6	2	16	15	10	13	12	14	5	7	3	9	4	11	8	1
6	15	16	14	10	13	12	2	5	7	3	9	4	11	8	1
16	15	13	14	10	9	12	2	5	7	3	6	4	11	8	1

Figure: An example of building a heap bottom up.

Source: adapted from [Manber 1989, Figure 6.15].

A Lower Bound for Sorting

- A lower bound for a particular problem is a proof that *no algorithm* can solve the problem better.
- We typically define a computation model and consider only those algorithms that fit in the model.
- **Decision trees** model computations performed by *comparison-based* algorithms.

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Theorem (Theorem 6.1)

Every decision-tree algorithm for sorting has height $\Omega(n \log n)$.

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Theorem (Theorem 6.1)

Every decision-tree algorithm for sorting has height $\Omega(n \log n)$.

Proof idea: there must be at least $n!$ leaves in the decision tree, one for each possible outcome.

A Lower Bound for Sorting

- A lower bound for a particular problem is a proof that *no algorithm* can solve the problem better.
- We typically define a computation model and consider only those algorithms that fit in the model.
- **Decision trees** model computations performed by *comparison-based* algorithms.

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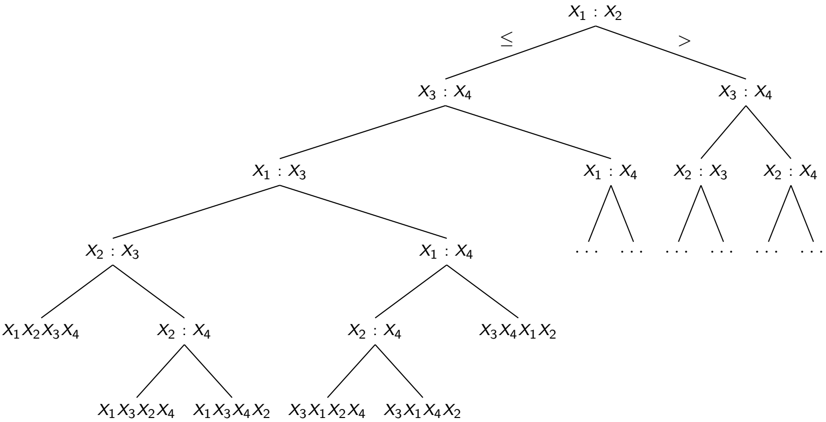
Is the lower bound contradictory to the time complexity of radix sort?

A Lower Bound for Sorting (cont.)

A decision tree (partly shown) for the merge sort with $X_1X_2X_3X_4$ as input:



Note: in total, there should be $4! = 24$ leaves, only six of which are shown.

Order Statistics: Minimum and Maximum

Problem

Find the maximum and minimum elements in a given sequence.

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- The obvious solution requires $(n-1)+(n-2)$ $(=2n-3)$ comparisons between elements.
- Can we do better? (Which comparisons could have been avoided?)

Order Statistics: Kth-Smallest

Problem

Given a sequence $S = x_1, x_2, \cdots, x_n$ of elements, and an integer k such that $1 \leq k \leq n$, find the kth-smallest element in S.

Order Statistics: *K*th-Smallest (cont.)

procedure Select $(Left, Right, k)$;
begin
  if $Left = Right$ **then**
    $Select := Left$
  else $Partition(X, Left, Right)$;
    let Middle be the output of Partition;
    if $Middle - Left + 1 \geq k$ **then**
      $Select(Left, Middle, k)$
    else
      $Select(Middle + 1, Right, k - (Middle - Left + 1))$
end

Algorithm Selection (X, n, k);
begin
  if $(k < 1)$ or $(k > n)$ **then** *print "error"*
  else $S := Select(1, n, k)$
end

Order Statistics: Kth-Smallest (cont.)

The nested "**if**" statement may be simplified:

procedure Select $(Left, Right, k)$;
begin
    if $Left = Right$ **then**
        $Select := Left$
    else $Partition(X, Left, Right)$;
        let Middle be the output of Partition;
        if $Middle \geq k$ **then**
            $Select(Left, Middle, k)$
        else
            $Select(Middle + 1, Right, k)$
end

Finding a Majority

Problem

Given a sequence of numbers, find the majority in the sequence or determine that none exists.

A number is a *majority* in a sequence if it occurs more than $\frac{n}{2}$ times in the sequence.

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Caution: maintaining a counter for each possible number requires $O(\log n)$ time for each access to a particular counter, which means $O(n \log n)$ time in total. Sorting the sequence to find a probable candidate also requires $O(n \log n)$ time.

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Idea: compare any two numbers in the sequence. What can we conclude if they are not equal?

What if they are equal?

Finding a Majority (cont.)

Algorithm Majority (X, n);
begin
    $C := X[1]$; $M := 1$;
    for $i := 2$ **to** n **do**
        if $M = 0$ **then**
            $C := X[i]$; $M := 1$
        else
            if $C = X[i]$ **then** $M := M + 1$
            else $M := M - 1$;

Finding a Majority (cont.)

if $M = 0$ **then** $Majority := -1$
else
  $Count := 0;$
  for $i := 1$ **to** n **do**
    if $X[i] = C$ **then** $Count := Count + 1;$
  if $Count > n/2$ **then** $Majority := C$
  else $Majority := -1$
end

Finding a Majority (cont.)

if $M = 0$ **then** $Majority := -1$
else
    $Count := 0$;
    for $i := 1$ **to** n **do**
        if $X[i] = C$ **then** $Count := Count + 1$;
    if $Count > n/2$ **then** $Majority := C$
    else $Majority := -1$
end

Time complexity: $O(n)$.