

# Soundness and Completeness of Hoare Logic

(Based on [Apt and Olderog 1997])

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# Overview

- Given an adequate semantics for the programming language under consideration, the **validity** of a Hoare triple  $\{p\} S \{q\}$  can be precisely defined.
- A Hoare Logic for a programming language is **sound** if every *Hoare triple proven by the logic is valid*.
- A Hoare Logic for a programming language is **complete** if every *valid Hoare triple can be proven by the logic*.
- We shall develop these results for a very simple **deterministic** programming language.

# A Simple Programming Language

- We will consider a Hoare Logic for the following simple (deterministic) programming language:

$$S ::= \begin{array}{l} \mathbf{skip} \\ | u := t \\ | S_1; S_2 \\ | \mathbf{if } B \mathbf{ then } S_1 \mathbf{ else } S_2 \mathbf{ fi} \\ | \mathbf{while } B \mathbf{ do } S \mathbf{ od} \end{array}$$

Note: here  $t$  is an expression (first-order term) of the same type as variable  $u$ ;  $B$  is a boolean expression.

- We consider only programs that are free of syntactical or typing errors.

# Proof Rules of Hoare Logic

$$\frac{}{\{q[t/u]\} u := t \{q\}}$$

(Assignment)

$$\frac{}{\{p\} \text{skip} \{p\}}$$

(Skip)

$$\frac{\{p\} S_1 \{q\} \quad \{q\} S_2 \{r\}}{\{p\} S_1; S_2 \{r\}}$$

(Sequence)

$$\frac{\{p \wedge B\} S_1 \{q\} \quad \{p \wedge \neg B\} S_2 \{q\}}{\{p\} \text{if } B \text{ then } S_1 \text{ else } S_2 \text{ fi} \{q\}}$$

(Conditional)

# Proof Rules of Hoare Logic (cont.)

$$\frac{\{p \wedge B\} S \{p\}}{\{p\} \mathbf{while} B \mathbf{do} S \mathbf{od} \{p \wedge \neg B\}} \quad \text{(While)}$$

$$\frac{p \rightarrow p' \quad \{p'\} S \{q'\} \quad q' \rightarrow q}{\{p\} S \{q\}} \quad \text{(Consequence)}$$

We will refer to this proof system as *System PD*.

# Operational Semantics

- 🌐 A program/statement with a start state is seen as an **abstract machine**.
- 🌐 (1) The **part of program that remains** to be executed and (2) the **current state** constitute the **configuration** of the abstract machine.
- 🌐 By executing the program step by step, the machine transforms from one configuration to another.
- 🌐 A **transition relation** naturally arises between configurations.
- 🌐 The (input/output) semantics  $\mathcal{M}[S]$  of a program  $S$  can then be defined with the help of the above transition relation.

# Operational Semantics (cont.)

🌐 At a high level, a configuration is a pair  $\langle S, \sigma \rangle$  where  $S$  is a program and  $\sigma$  is a “proper” state.

🌐 A transition

$$\langle S, \sigma \rangle \rightarrow \langle R, \tau \rangle$$

means “executing  $S$  one step in state  $\sigma$  leads to state  $\tau$  with  $R$  as the remainder of  $S$  to be executed.”

🌐 Let  $E$  denote the empty program. When the remainder  $R$  equals  $E$ , it means that  $S$  has terminated.

🌐 The transition relation  $\rightarrow$  can be defined inductively (in the form of axioms and rules) over the structure of a program.

# Semantics of the Simple Language

To give an **operational semantics** of the simple language, we postulate the following **transition axioms** and **rules**:

1.  $\langle \mathbf{skip}, \sigma \rangle \rightarrow \langle E, \sigma \rangle$
2.  $\langle u := t, \sigma \rangle \rightarrow \langle E, \sigma[u := \sigma(t)] \rangle$
3. 
$$\frac{\langle S_1, \sigma \rangle \rightarrow \langle S_2, \tau \rangle}{\langle S_1; S, \sigma \rangle \rightarrow \langle S_2; S, \tau \rangle}$$
4.  $\langle \mathbf{if } B \mathbf{ then } S_1 \mathbf{ else } S_2 \mathbf{ fi}, \sigma \rangle \rightarrow \langle S_1, \sigma \rangle$ , when  $\sigma \models B$
5.  $\langle \mathbf{if } B \mathbf{ then } S_1 \mathbf{ else } S_2 \mathbf{ fi}, \sigma \rangle \rightarrow \langle S_2, \sigma \rangle$ , when  $\sigma \models \neg B$
6.  $\langle \mathbf{while } B \mathbf{ do } S \mathbf{ od}, \sigma \rangle \rightarrow \langle S; \mathbf{while } B \mathbf{ do } S \mathbf{ od}, \sigma \rangle$ , when  $\sigma \models B$
7.  $\langle \mathbf{while } B \mathbf{ do } S \mathbf{ od}, \sigma \rangle \rightarrow \langle E, \sigma \rangle$ , when  $\sigma \models \neg B$



- 🌐 The preceding set of transition axioms and rules can be seen as a formal proof system, called a **transition system**.
- 🌐 A transition  $\langle S, \sigma \rangle \rightarrow \langle R, \tau \rangle$  is possible if it can be **deduced** in the transition system.
- 🌐 This semantic is “high level”, as assignments and evaluations of Boolean expressions are done in one step.

- A *transition sequence* of  $S$  starting in  $\sigma$  is a finite or infinite sequence of configurations

$$\langle S_0, \sigma_0 \rangle (= \langle S, \sigma \rangle) \rightarrow \langle S_1, \sigma_1 \rangle \rightarrow \cdots \rightarrow \langle S_i, \sigma_i \rangle \rightarrow \cdots$$

- A *computation* of  $S$  starting in  $\sigma$  is a transition sequence of  $S$  starting in  $\sigma$  that cannot be extended.
- A computation of  $S$  *terminates* in  $\tau$  if it is finite and its last configuration is  $\langle E, \tau \rangle$ .
- A computation of  $S$  *diverges* if it is infinite.

# An Example

- Consider the following program

$$S \equiv a[0] := 1; a[1] := 0; \mathbf{while} \ a[x] \neq 0 \ \mathbf{do} \ x := x + 1 \ \mathbf{od}$$

- Let  $\sigma$  be a state in which  $x$  is 0.
- Let  $\sigma'$  stand for  $\sigma[a[0] := 1][a[1] := 0]$ .
- The following is the computation of  $S$  starting in  $\sigma$ :

$$\begin{aligned} & \langle S, \sigma \rangle \\ \rightarrow & \langle a[1] := 0; \mathbf{while} \ a[x] \neq 0 \ \mathbf{do} \ x := x + 1 \ \mathbf{od}, \sigma[a[0] := 1] \rangle \\ \rightarrow & \langle \mathbf{while} \ a[x] \neq 0 \ \mathbf{do} \ x := x + 1 \ \mathbf{od}, \sigma' \rangle \\ \rightarrow & \langle x := x + 1; \mathbf{while} \ a[x] \neq 0 \ \mathbf{do} \ x := x + 1 \ \mathbf{od}, \sigma' \rangle \\ \rightarrow & \langle \mathbf{while} \ a[x] \neq 0 \ \mathbf{do} \ x := x + 1 \ \mathbf{od}, \sigma'[x := 1] \rangle \\ \rightarrow & \langle E, \sigma'[x := 1] \rangle \end{aligned}$$

# Finite Transition Sequences

- For partial correctness of sequential programs, we will need only to talk about **finite** transition sequences.
- To that end, we take the **reflexive transitive closure**  $\rightarrow^*$  of  $\rightarrow$ .
- So,  $\langle S, \sigma \rangle \rightarrow^* \langle R, \tau \rangle$  holds when
  - $\langle R, \tau \rangle = \langle S, \sigma \rangle$  or
  - $\langle S_0, \sigma_0 \rangle (= \langle S, \sigma \rangle) \rightarrow \langle S_1, \sigma_1 \rangle \rightarrow \cdots \rightarrow \langle S_n, \sigma_n \rangle (= \langle R, \tau \rangle)$  is a finite transition sequence.

# Input/Output Semantics

- Let  $\Sigma$  be the set of all “proper” states.
- The *partial correctness semantics* is a mapping

$$\mathcal{M}[[S]] : \Sigma \rightarrow \mathcal{P}(\Sigma)$$

with

$$\mathcal{M}[[S]](\sigma) = \{\tau \mid \langle S, \sigma \rangle \rightarrow^* \langle E, \tau \rangle\}.$$

- Extensions of  $\mathcal{M}[[S]]$ 
  - $\mathcal{M}[[S]](\perp) = \emptyset.$
  - For  $X \subseteq \Sigma \cup \{\perp\}$ ,  $\mathcal{M}[[S]](X) = \bigcup_{\sigma \in X} \mathcal{M}[[S]](\sigma).$

# Validity of a Hoare Triple

- Let  $\llbracket p \rrbracket$  denote  $\{\sigma \in \Sigma \mid \sigma \models p\}$ , i.e., the set of states where  $p$  holds.
- The Hoare triple  $\{p\} S \{q\}$  is **valid** in the sense of partial correctness, written  $\models \{p\} S \{q\}$ , if

$$\mathcal{M}[S](\llbracket p \rrbracket) \subseteq \llbracket q \rrbracket.$$

# About the While Loop

- Let  $\Omega$  be a program such that  $\mathcal{M}[\Omega](\sigma) = \emptyset$ , for any  $\sigma$ .
- Define the following sequence of deterministic programs:

$$\begin{aligned} (\mathbf{while\ } B \mathbf{\ do\ } S \mathbf{\ od})^0 &= \Omega \\ (\mathbf{while\ } B \mathbf{\ do\ } S \mathbf{\ od})^{k+1} &= \mathbf{if\ } B \mathbf{\ then\ } S; (\mathbf{while\ } B \mathbf{\ do\ } S \mathbf{\ od})^k \\ &\quad \mathbf{else\ skip\ fi} \end{aligned}$$

- For example,  $(\mathbf{while\ } B \mathbf{\ do\ } S \mathbf{\ od})^2$ 
  - $= \mathbf{if\ } B \mathbf{\ then\ } S; (\mathbf{while\ } B \mathbf{\ do\ } S \mathbf{\ od})^1$   
 $\mathbf{else\ skip\ fi}$
  - $= \mathbf{if\ } B \mathbf{\ then\ } S; \mathbf{if\ } B \mathbf{\ then\ } S; (\mathbf{while\ } B \mathbf{\ do\ } S \mathbf{\ od})^0$   
 $\mathbf{else\ skip\ fi}$
  - $= \mathbf{if\ } B \mathbf{\ then\ } S; \mathbf{if\ } B \mathbf{\ then\ } S; \Omega$   
 $\mathbf{else\ skip\ fi}$

# Lemmas for $\mathcal{M}[[S]]$

1.  $\mathcal{M}[[S]]$  is **monotonic**, i.e.,  
 $X \subseteq Y \subseteq \Sigma \cup \{\perp\}$  implies  $\mathcal{M}[[S]](X) \subseteq \mathcal{M}[[S]](Y)$ .
2.  $\mathcal{M}[[S_1; S_2]](X) = \mathcal{M}[[S_2]](\mathcal{M}[[S_1]](X))$ .
3.  $\mathcal{M}[[S_1; S_2]; S_3]](X) = \mathcal{M}[[S_1; (S_2; S_3)]](X)$ .
4.  $\mathcal{M}[[\text{if } B \text{ then } S_1 \text{ else } S_2 \text{ fi}]](X) =$   
 $\mathcal{M}[[S_1]](X \cap [B]) \cup \mathcal{M}[[S_2]](X \cap [\neg B])$ .
5.  $\mathcal{M}[[\text{while } B \text{ do } S \text{ od}]] = \bigcup_{k=0}^{\infty} \mathcal{M}[[\text{(while } B \text{ do } S \text{ od)}^k]]$ .



**Theorem** (Soundness): The proof system  $PD$  is sound for partial correctness of programs in the simple programming language, i.e.,

$$\vdash_{PD} \{p\} S \{q\} \text{ implies } \models \{p\} S \{q\}.$$

The proof is by induction, i.e., by proving that (1) the Hoare triples in all axioms of  $PD$  are valid and (2) all proof rules of  $PD$  are sound.

Note: a proof rule is sound if the validity of the Hoare triples in the premises implies the validity of the Hoare triple in the conclusion.

# Soundness (cont.)

🌐 **skip:**  $\mathcal{M}[\mathbf{skip}](\llbracket p \rrbracket) \subseteq \llbracket p \rrbracket$

$$\begin{aligned} \mathcal{M}[\mathbf{skip}](\llbracket p \rrbracket) &= \bigcup_{\sigma \in \llbracket p \rrbracket} \{ \tau \mid \langle \mathbf{skip}, \sigma \rangle \rightarrow^* \langle E, \tau \rangle \} \\ &= \bigcup_{\sigma \in \llbracket p \rrbracket} \{ \sigma \} = \llbracket p \rrbracket \subseteq \llbracket p \rrbracket. \end{aligned}$$

🌐 **Assignment:**  $\mathcal{M}[u := t](\llbracket p[t/u] \rrbracket) \subseteq \llbracket p \rrbracket$

It can be shown that (1)  $\sigma(s[u := t]) = \sigma[u := \sigma(t)](s)$  and (2)  $\sigma \models p[t/u]$  iff  $\sigma[u := \sigma(t)] \models p$ .

Let  $\sigma \in \llbracket p[t/u] \rrbracket$ .

From the transition axiom for assignment,

$$\mathcal{M}[u := t](\sigma) = \{ \sigma[u := \sigma(t)] \}.$$

Since  $\sigma \models p[t/u]$  iff  $\sigma[u := \sigma(t)] \models p$ , we have

$$\mathcal{M}[u := t](\sigma) \subseteq \llbracket p \rrbracket \text{ and hence } \mathcal{M}[u := t](\llbracket p[t/u] \rrbracket) \subseteq \llbracket p \rrbracket.$$

## Soundness (cont.)

- Composition:  $\mathcal{M}[S_1](\llbracket p \rrbracket) \subseteq \llbracket r \rrbracket$  and  $\mathcal{M}[S_2](\llbracket r \rrbracket) \subseteq \llbracket q \rrbracket$  imply  $\mathcal{M}[S_1; S_2](\llbracket p \rrbracket) \subseteq \llbracket q \rrbracket$ .

From the monotonicity of  $\mathcal{M}[S_2]$ ,  
 $\mathcal{M}[S_2](\mathcal{M}[S_1](\llbracket p \rrbracket)) \subseteq \mathcal{M}[S_2](\llbracket r \rrbracket) \subseteq \llbracket q \rrbracket$ .

By an earlier lemma,  $\mathcal{M}[S_2](\mathcal{M}[S_1](\llbracket p \rrbracket)) = \mathcal{M}[S_1; S_2](\llbracket p \rrbracket)$ .

- Conditional:  $\mathcal{M}[S_1](\llbracket p \wedge B \rrbracket) \subseteq \llbracket q \rrbracket$  and  $\mathcal{M}[S_2](\llbracket p \wedge \neg B \rrbracket) \subseteq \llbracket q \rrbracket$  imply  $\mathcal{M}[\mathbf{if } B \mathbf{ then } S_1 \mathbf{ else } S_2 \mathbf{ fi}](\llbracket p \rrbracket) \subseteq \llbracket q \rrbracket$ .

This follows from an earlier lemma,  
 $\mathcal{M}[\mathbf{if } B \mathbf{ then } S_1 \mathbf{ else } S_2 \mathbf{ fi}](X) = \mathcal{M}[S_1](X \cap \llbracket B \rrbracket) \cup \mathcal{M}[S_2](X \cap \llbracket \neg B \rrbracket)$ .

## Soundness (cont.)

- While:  $\mathcal{M}[[S]](\llbracket p \wedge B \rrbracket) \subseteq \llbracket p \rrbracket$  implies  $\mathcal{M}[\mathbf{while} B \mathbf{do} S \mathbf{od}](\llbracket p \rrbracket) \subseteq \llbracket p \wedge \neg B \rrbracket$ .

From Lemma 5 for  $\mathcal{M}[\cdot]$ , it boils down to show that  $\bigcup_{k=0}^{\infty} \mathcal{M}[(\mathbf{while} B \mathbf{do} S \mathbf{od})^k](\llbracket p \rrbracket) \subseteq \llbracket p \wedge \neg B \rrbracket$ .

We prove by induction that, for all  $k \geq 0$ ,

$$\mathcal{M}[(\mathbf{while} B \mathbf{do} S \mathbf{od})^k](\llbracket p \rrbracket) \subseteq \llbracket p \wedge \neg B \rrbracket.$$

The base case  $k = 0$  is clear.

# Soundness (cont.)

$$\begin{aligned}
 & \mathcal{M}[\langle \mathbf{while} \ B \ \mathbf{do} \ S \ \mathbf{od} \rangle^{k+1}]([\![p]\!]) \\
 = & \quad \{ \text{definition of } \langle \mathbf{while} \ B \ \mathbf{do} \ S \ \mathbf{od} \rangle^{k+1} \} \\
 & \mathcal{M}[\langle \mathbf{if} \ B \ \mathbf{then} \ S; \langle \mathbf{while} \ B \ \mathbf{do} \ S \ \mathbf{od} \rangle^k \ \mathbf{else} \ \mathbf{skip} \ \mathbf{fi} \rangle]([\![p]\!]) \\
 = & \quad \{ \text{Lemma 4 for } \mathcal{M}[\cdot] \} \\
 & \mathcal{M}[S; \langle \mathbf{while} \ B \ \mathbf{do} \ S \ \mathbf{od} \rangle^k]([\![p \wedge B]\!]) \cup \mathcal{M}[\mathbf{skip}]([\![p \wedge \neg B]\!]) \\
 = & \quad \{ \text{Lemma 2 for } \mathcal{M}[\cdot] \text{ and semantics of } \mathbf{skip} \} \\
 & \mathcal{M}[\langle \mathbf{while} \ B \ \mathbf{do} \ S \ \mathbf{od} \rangle^k](\mathcal{M}[S]([\![p \wedge B]\!]) \cup [\![p \wedge \neg B]\!]) \\
 \subseteq & \quad \{ \text{the premise and monotonicity of } \mathcal{M}[\cdot] \} \\
 & \mathcal{M}[\langle \mathbf{while} \ B \ \mathbf{do} \ S \ \mathbf{od} \rangle^k]([\![p]\!]) \cup [\![p \wedge \neg B]\!] \\
 \subseteq & \quad \{ \text{induction hypothesis} \} \\
 & [\![p \wedge \neg B]\!] \cup [\![p \wedge \neg B]\!]
 \end{aligned}$$

🌐 Consequence:  $p \rightarrow p'$ ,  $\mathcal{M}[S](\llbracket p' \rrbracket) \subseteq \llbracket q' \rrbracket$ , and  $q' \rightarrow q$  imply  $\mathcal{M}[S](\llbracket p \rrbracket) \subseteq \llbracket q \rrbracket$ .

First of all,  $\llbracket p \rrbracket \subseteq \llbracket p' \rrbracket$  and  $\llbracket q' \rrbracket \subseteq \llbracket q \rrbracket$ .

From the monotonicity of  $\mathcal{M}[S]$ ,  
 $\mathcal{M}[S](\llbracket p \rrbracket) \subseteq \mathcal{M}[S](\llbracket p' \rrbracket) \subseteq \llbracket q' \rrbracket \subseteq \llbracket q \rrbracket$ .

# About Completeness

- 🌐 Assertions that we use for a programming language often involve numbers/integers.
- 🌐 According to **Gödel's First Incompleteness Theorem**, there is no complete proof system (that is consistent/sound) for the first-order theory of arithmetic.
- 🌐 We therefore assume that all true assertions are given (as axioms).
- 🌐 The completeness of Hoare Logic then is actually **relative to the truth of all assertions**.

# Weakest Liberal Precondition

- Let  $S$  be a program in the simple programming language.
- For a set  $\Phi$  of states, we define

$$wlp(S, \Phi) = \{\sigma \mid \mathcal{M}[[S]](\sigma) \subseteq \Phi\}.$$

- $wlp(S, \Phi)$  is called the *weakest liberal precondition* of  $S$  with respect to  $\Phi$ .
- Informally,  $wlp(S, \Phi)$  is the set of all states  $\sigma$  such that whenever  $S$  is activated in  $\sigma$  and properly terminates, the output state is in  $\Phi$ .



# Definability of $wlp(S, \Phi)$

- 🌍 An assertion  $p$  defines a set  $\Phi$  of states if  $\llbracket p \rrbracket = \Phi$ .
- 🌍 Assuming that the assertion language includes addition and multiplication of natural numbers,  
*there is an assertion  $p$  defining  $wlp(S, \llbracket q \rrbracket)$ , i.e., with  $\llbracket p \rrbracket = wlp(S, \llbracket q \rrbracket)$ .*
- 🌍 Proof of the above statement requires a technique called *Gödelization* and will not be given here.
- 🌍 We will write  $wlp(S, q)$  to denote the assertion  $p$  such that  $\llbracket p \rrbracket = wlp(S, \llbracket q \rrbracket)$ .

# Lemmas for $wlp$

1.  $wlp(\mathbf{skip}, q) \leftrightarrow q$ .
2.  $wlp(u := t, q) \leftrightarrow q[t/u]$ .
3.  $wlp(S_1; S_2, q) \leftrightarrow wlp(S_1, wlp(S_2, q))$ .
4.  $wlp(\mathbf{if } B \mathbf{ then } S_1 \mathbf{ else } S_2 \mathbf{ fi}, q) \leftrightarrow (B \wedge wlp(S_1, q)) \vee (\neg B \wedge wlp(S_2, q))$ .
5.  $wlp(\mathbf{while } B \mathbf{ do } S_1 \mathbf{ od}, q) \wedge B \rightarrow wlp(S_1, wlp(\mathbf{while } B \mathbf{ do } S_1 \mathbf{ od}, q))$ .
6.  $wlp(\mathbf{while } B \mathbf{ do } S_1 \mathbf{ od}, q) \wedge \neg B \rightarrow q$ .
7.  $\models \{p\} S \{q\}$  iff  $p \rightarrow wlp(S, q)$ .

# Completeness

**Theorem** (Completeness): The proof system  $PD$  is complete for partial correctness of programs in the simple programming language, i.e.,

$$\models \{p\} S \{q\} \text{ implies } \vdash_{PD} \{p\} S \{q\}.$$

- 🌍 We first prove a special case of the theorem: for all  $S$  and  $q$ ,  $\models \{wlp(S, q)\} S \{q\}$  implies  $\vdash_{PD} \{wlp(S, q)\} S \{q\}$ .
- 🌍 As  $\models \{wlp(S, q)\} S \{q\}$  (i.e.,  $\mathcal{M}[\![S]\!](\![wlp(S, q)]\!) \subseteq \![q]\!)$  always holds, the case simplifies to  $\vdash_{PD} \{wlp(S, q)\} S \{q\}$ , for all  $S$  and  $q$ .
- 🌍 This is done by **induction**.
- 🌍 The base cases (**skip** and assignment) are trivial.

## Completeness (cont.)

🌐 Conditional:  $S \equiv \text{if } B \text{ then } S_1 \text{ else } S_2 \text{ fi}$ .

To prove  $\vdash_{PD} \{wlp(S, q)\} S \{q\}$  via the conditional rule, we need

- (1)  $\vdash_{PD} \{wlp(S, q) \wedge B\} S_1 \{q\}$  and
- (2)  $\vdash_{PD} \{wlp(S, q) \wedge \neg B\} S_2 \{q\}$ .

From the induction hypothesis, we have

- (3)  $\vdash_{PD} \{wlp(S_1, q)\} S_1 \{q\}$  and
- (4)  $\vdash_{PD} \{wlp(S_2, q)\} S_2 \{q\}$ .

Applying the consequence rule, we are done if

- (5)  $wlp(S, q) \wedge B \rightarrow wlp(S_1, q)$  and
- (6)  $wlp(S, q) \wedge \neg B \rightarrow wlp(S_2, q)$ .

(5) and (6) follows from Lemma 4 for  $wlp$ .

## Completeness (cont.)

A proof of (5)  $wlp(S, q) \wedge B \rightarrow wlp(S_1, q)$ :

$$\begin{aligned}
& wlp(S, q) \wedge B \\
\leftrightarrow & \{ \text{definition of } S \} \\
& wlp(\mathbf{if } B \mathbf{ then } S_1 \mathbf{ else } S_2 \mathbf{ fi}, q) \wedge B \\
\leftrightarrow & \{ \text{Lemma 4 for } wlp \} \\
& ((B \wedge wlp(S_1, q)) \vee (\neg B \wedge wlp(S_2, q))) \wedge B \\
\leftrightarrow & \{ \text{distribute } \wedge \text{ over } \vee \} \\
& ((B \wedge wlp(S_1, q)) \wedge B) \vee ((\neg B \wedge wlp(S_2, q)) \wedge B) \\
\leftrightarrow & \{ \text{commutativity of } \wedge, B \wedge B \leftrightarrow B, \text{ and } \neg B \wedge B \leftrightarrow \text{false} \} \\
& (B \wedge wlp(S_1, q)) \vee \text{false} \\
\leftrightarrow & \{ A \vee \text{false} \leftrightarrow A \} \\
& B \wedge wlp(S_1, q) \\
\rightarrow & \{ B \wedge A \rightarrow A \} \\
& wlp(S_1, q)
\end{aligned}$$

## Completeness (cont.)

🌐 While:  $S \equiv \mathbf{while\ } B \mathbf{\ do\ } S_1 \mathbf{\ od.}$

To prove  $\vdash_{PD} \{wlp(S, q)\} S \{q\}$ , we apply Lemma 6 for  $wlp$  and the consequence rule to reduce the goal to  $\vdash_{PD} \{wlp(S, q)\} S \{wlp(S, q) \wedge \neg B\}$ .

Using the while rule, the goal is further reduced to  $\vdash_{PD} \{wlp(S, q) \wedge B\} S_1 \{wlp(S, q)\}$ .

From Lemma 5 for  $wlp$  and the consequence rule, we need  $\vdash_{PD} \{wlp(S_1, wlp(S, q))\} S_1 \{wlp(S, q)\}$ .

This follows from the induction hypothesis, which states that  $\vdash_{PD} \{wlp(S_1, q')\} S_1 \{q'\}$ , for all  $q'$ .

# Completeness (cont.)

- Now suppose  $\models \{p\} S \{q\}$ .
- From Lemma 7 for  $wlp$ ,  $p \rightarrow wlp(S, q)$ .
- From  $\vdash_{PD} \{wlp(S, q)\} S \{q\}$  and the consequence rule,  $\vdash_{PD} \{p\} S \{q\}$ .