Suggested Solutions for Homework Assignment #1

We assume the binding powers of the logical connectives and the entailment symbol decrease in this order: $\neg$, $\{\wedge, \vee\}$, $\rightarrow$, $\leftrightarrow$, $\vdash$.

1. (30 points) Prove that every propositional formula has an equivalent formula in the conjunctive normal form and also an equivalent formula in the disjunctive normal form. (Hint: by induction on the structure of a formula, dealing with both cases simultaneously)

   Solution. Let us first review some preliminaries. A (non-empty) clause is a disjunction of one or more literals such as $p \vee \neg q \vee r$, while a (non-empty) term/product is a conjunction of one or more literals such as $\neg p \wedge q \wedge \neg r$. (Note: the name "term" as defined here is not commonly used in propositional logic. However, it is adequate in light of the notion of a term in algebraic expressions. An alternative name is "product".) So, a formula is in conjunctive normal form (CNF) if it is a conjunction of one or more clauses. A formula is in disjunctive normal form (DNF) if it is a disjunction of one or more terms. A clause by itself is in CNF (a one-clause CNF) and, when seen as a disjunction of one-literal terms, is also in DNF. Similarly, for a term. A single literal is a special case of a clause and also of a term.

   The complement of a clause (term), after the negation is pushed to the literal level, becomes a term (clause), e.g., $\neg(p \vee \neg q \vee r) \Leftrightarrow \neg p \wedge q \wedge \neg r$. Taking this one level up, the complement of a formula in CNF (DNF), after the negation is pushed to the literal level, becomes a formula in DNF (CNF), e.g., $\neg((p \vee \neg q) \wedge (q \vee r)) \Leftrightarrow (\neg p \wedge q) \vee (\neg q \wedge \neg r)$.

   Now we prove the problem statement by induction on the structure of a given formula φ.

   Base case (φ is just a propositional symbol): a propositional symbol can be seen as a single-literal clause or term and so is already in CNF and in DNF.

   Inductive step: there are three cases.

   (a) $\varphi = \neg\psi$: let ψ^C be a formula equivalent to ψ in CNF and ψ^D an equivalent formula in DNF (guaranteed to exist by the induction hypothesis). Pushing the negation at the front of $\neg\psi^C$ ($\neg\psi^D$) to the literal level, we get a formula equivalent to φ in DNF (CNF).

   (b) $\varphi = \varphi_1 \wedge \varphi_2$: let φ_1^C (φ_2^C) be a formula equivalent to φ_1 (φ_2) in CNF and φ_1^D (φ_2^D) an equivalent formula in DNF. The formula $\varphi_1^C \wedge \varphi_2^C$ is equivalent to φ and readily in CNF.

   To obtain a formula equivalent to φ in DNF, suppose $\varphi_1^D = t_1 \vee t_2 \vee \cdots \vee t_l$ and $\varphi_2^D = u_1 \vee u_2 \vee \cdots \vee u_m$, where t_i's and u_j's are terms. Then, by repeatedly distributing the top-level $\wedge$ in $\varphi_1^D \wedge \varphi_2^D$ to the term level, we obtain a formula $\bigvee_{1 \leq i \leq l, 1 \leq j \leq m}(t_i \wedge u_j)$ in DNF that is equivalent to φ.

   (c) $\varphi = \varphi_1 \vee \varphi_2$: analogous to the case of $\varphi = \varphi_1 \wedge \varphi_2$.

   □

2. (40 points) Prove, using *Natural Deduction* (in the sequent form), the validity of the following sequents:

(a) $p \vee q \to r \vdash (p \to r) \wedge (q \to r)$
Solution.

$$\dfrac{\dfrac{\alpha}{p \vee q \to r \vdash p \to r}\,(\to I) \qquad \dfrac{\beta}{p \vee q \to r \vdash q \to r}\,(\to I)}{p \vee q \to r \vdash (p \to r) \wedge (q \to r)}\,(\wedge I)$$

α :

$$\dfrac{\dfrac{}{p \vee q \to r, p \vdash p \vee q \to r}\,(Hyp) \qquad \dfrac{\dfrac{}{p \vee q \to r, p \vdash p}\,(Hyp)}{p \vee q \to r, p \vdash p \vee q}\,(\vee I_1)}{p \vee q \to r, p \vdash r}\,(\to E)$$

β :

$$\dfrac{\dfrac{}{p \vee q \to r, q \vdash p \vee q \to r}\,(Hyp) \qquad \dfrac{\dfrac{}{p \vee q \to r, q \vdash q}\,(Hyp)}{p \vee q \to r, q \vdash p \vee q}\,(\vee I_2)}{p \vee q \to r, q \vdash r}\,(\to E)$$

$\square$

(b) $\vdash (p \to (q \to r)) \to (p \wedge q \to r)$
Solution.

$$\dfrac{\dfrac{\dfrac{\alpha \qquad \dfrac{\dfrac{}{p \to (q \to r), p \wedge q \vdash p \wedge q}\,(Hyp)}{p \to (q \to r), p \wedge q \vdash q}\,(\wedge E_2)}{p \to (q \to r), p \wedge q \vdash r}\,(\to E)}{p \to (q \to r) \vdash p \wedge q \to r}\,(\to I)}{\vdash (p \to (q \to r)) \to (p \wedge q \to r)}\,(\to I)$$

α :

$$\dfrac{\dfrac{}{p \to (q \to r), p \wedge q \vdash p \to (q \to r)}\,(Hyp) \qquad \dfrac{\dfrac{}{p \to (q \to r), p \wedge q \vdash p \wedge q}\,(Hyp)}{p \to (q \to r), p \wedge q \vdash p}\,(\wedge E_1)}{p \to (q \to r), p \wedge q \vdash q \to r}\,(\to E)$$

$\square$

3. (30 points) Prove, using *Natural Deduction* (in the sequent form), the validity of the following sequents:

   (a) $\vdash (p \to q) \to (\neg p \vee q)$
   Solution.

$$
\begin{prooftree}
\AxiomC{α}
\AxiomC{}
\RightLabel{(Hyp)}
\UnaryInfC{$p \to q, \neg(\neg p \vee q), p \vdash \neg(\neg p \vee q)$}
\RightLabel{$(\wedge I)$}
\BinaryInfC{$p \to q, \neg(\neg p \vee q), p \vdash (\neg p \vee q) \wedge \neg(\neg p \vee q)$}
\RightLabel{$(\neg I)$}
\UnaryInfC{$p \to q, \neg(\neg p \vee q) \vdash \neg p$}
\RightLabel{$(\vee I_1)$}
\UnaryInfC{$p \to q, \neg(\neg p \vee q) \vdash \neg p \vee q$}
\AxiomC{}
\RightLabel{(Hyp)}
\UnaryInfC{$p \to q, \neg(\neg p \vee q) \vdash \neg(\neg p \vee q)$}
\RightLabel{$(\wedge I)$}
\BinaryInfC{$p \to q, \neg(\neg p \vee q) \vdash (\neg p \vee q) \wedge \neg(\neg p \vee q)$}
\RightLabel{$(\neg I)$}
\UnaryInfC{$p \to q \vdash \neg\neg(\neg p \vee q)$}
\RightLabel{$(\neg\neg E)$}
\UnaryInfC{$p \to q \vdash \neg p \vee q$}
\RightLabel{$(\to I)$}
\UnaryInfC{$\vdash (p \to q) \to (\neg p \vee q)$}
\end{prooftree}
$$

$\alpha :$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Hyp)}
\UnaryInfC{$p \to q, \neg(\neg p \vee q), p \vdash p \to q$}
\AxiomC{}
\RightLabel{(Hyp)}
\UnaryInfC{$p \to q, \neg(\neg p \vee q), p \vdash p$}
\RightLabel{$(\to E)$}
\BinaryInfC{$p \to q, \neg(\neg p \vee q), p \vdash q$}
\RightLabel{$(\vee I_2)$}
\UnaryInfC{$p \to q, \neg(\neg p \vee q), p \vdash \neg p \vee q$}
\end{prooftree}
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□

(b) $\vdash ((p \to q) \to p) \to p$

Solution.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Hyp)}
\UnaryInfC{$(p \to q) \to p, \neg p \vdash (p \to q) \to p$}
\AxiomC{α}
\RightLabel{$(\to E)$}
\BinaryInfC{$(p \to q) \to p, \neg p \vdash p$}
\AxiomC{}
\RightLabel{(Hyp)}
\UnaryInfC{$(p \to q) \to p, \neg p \vdash \neg p$}
\RightLabel{$(\wedge I)$}
\BinaryInfC{$(p \to q) \to p, \neg p \vdash p \wedge \neg p$}
\RightLabel{$(\neg I)$}
\UnaryInfC{$(p \to q) \to p \vdash \neg\neg p$}
\RightLabel{$(\neg\neg E)$}
\UnaryInfC{$(p \to q) \to p \vdash p$}
\RightLabel{$(\to I)$}
\UnaryInfC{$\vdash ((p \to q) \to p) \to p$}
\end{prooftree}
$$

$\alpha :$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Hyp)}
\UnaryInfC{$(p \to q) \to p, \neg p, p, \neg q \vdash p$}
\AxiomC{}
\RightLabel{(Hyp)}
\UnaryInfC{$(p \to q) \to p, \neg p, p, \neg q \vdash \neg p$}
\RightLabel{$(\neg E)$}
\BinaryInfC{$(p \to q) \to p, \neg p, p \vdash q$}
\RightLabel{$(\to I)$}
\UnaryInfC{$(p \to q) \to p, \neg p \vdash p \to q$}
\end{prooftree}
$$

□