Final

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. Give a context-free grammar that generates the following language: $\{w \in \{a, b, c\}^* \mid$ the number of a's in w equals that of b's or c's$\}$ (no restriction is imposed on the order in which the input symbols may appear). Please make the CFG as simple as possible and explain the intuition behind it.

2. Give a formal description of a (single-tape deterministic) Turing machine that decides the language $\{1^k\#1^{2^k} \mid k \geq 1\}$.

3. Show that (single-tape) Turing machines which are allowed to move its head only to the right are less powerful than the usual Turing machines. What class of languages do this type of restricted Turing machines recognize? Please sketch a proof.

4. Let $CONTAIN_{\mathrm{PDA\_DFA}} = \{\langle M_1, M_2\rangle \mid M_1$ is a PDA and M_2 is a DFA such that $L(M_1) \subseteq (M_2)\}$. Show that $CONTAIN_{\mathrm{PDA\_DFA}}$ is decidable.

5. Let $CONTAIN_{\mathrm{DFA\_PDA}} = \{\langle M_1, M_2\rangle \mid M_1$ is a DFA and M_2 is a PDA such that $L(M_1) \subseteq (M_2)\}$. Show that $CONTAIN_{\mathrm{DFA\_PDA}}$ is undecidable.

6. Let $AMBIG_{\mathrm{CFG}} = \{\langle G\rangle \mid G$ is an ambiguous CFG$\}$. Show that $AMBIG_{\mathrm{CFG}}$ is undecidable. (Hint: use a reduction from PCP. Given an instance

$$P = \left\{ \left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots, \left[\frac{t_k}{b_k}\right] \right\}$$

of PCP, construct a CFG G with the rules:

$$\begin{array}{lcl} S & \to & T \mid B \\ T & \to & t_1Ta_1 \mid \cdots \mid t_kTa_k \mid t_1a_1 \mid \cdots \mid t_ka_k \\ B & \to & b_1Ba_1 \mid \cdots \mid b_kBa_k \mid b_1a_1 \mid \cdots \mid b_ka_k, \end{array}$$

where $a_1, \ldots, a_k$ are new terminal symbols. Prove that this reduction works.)

7. Show that if A is Turing-recognizable and $A \leq_m \overline{A}$, then A is decidable.

8. Discuss the applicability of Rice's Theorem for each of the following problems (languages). Please give the reasons why or why not.

   (a) $REGULAR_{\mathrm{TM}} = \{\langle M\rangle \mid M$ is a TM and $L(M)$ is regular$\}$.

(b) $W\_USELESS_{\mathrm{TM}} = \{\langle M \rangle \mid M$ is a TM with useless states$\}$. (A *useless state* in a Turing machine is one that is never entered on any input string.)

9. In the proof that the *3SAT* problem is polynomially reducible to the *VERTEX_COVER* problem, we convert an arbitrary boolean expression in 3CNF (input of the *3SAT* problem) to an input graph of the *VERTEX_COVER* problem.

   (a) Please illustrate the conversion by drawing the graph that will be obtained from the following boolean expression:

   $$(x + \overline{y} + z) \cdot (w + y + \overline{z}) \cdot (\overline{w} + x + \overline{y}).$$

   (b) The original boolean expression is satisfiable. As a demonstration of why the reduction is correct, please use the resulting graph to argue that it is indeed the case.

10. In the proof that the *3SAT* problem is polynomially reducible to the *SUBSET_SUM* problem, we convert an arbitrary boolean expression in 3CNF (input of the *3SAT* problem) to a set of numbers and a target number as input of the *SUBSET_SUM* problem.

    (a) Please illustrate the conversion by giving the set of numbers and the target number that will be obtained from the following boolean expression:

    $$(x + \overline{y} + z) \cdot (w + y + \overline{z}) \cdot (\overline{w} + x + \overline{y}).$$

    (b) The original boolean expression is satisfiable. As a demonstration of why the reduction is correct, please use the resulting set of numbers and the target number to argue that it is indeed the case.

Appendix

- $ALL_{\mathrm{CFG}} = \{\langle G \rangle \mid G$ is a CFG and $L(G) = \Sigma^*\}$. ALL_{CFG} is undecidable.
- **Rice's Theorem** states that any problem P about Turing machines satisfying the following two conditions is undecidable:
  1. For any TMs M_1 and M_2, where $L(M_1) = L(M_2)$, we have $\langle M_1 \rangle \in P$ iff $\langle M_2 \rangle \in P$.
  2. P is nontrivial, i.e., there exist TMs M_1 and M_2 such that $\langle M_1 \rangle \in P$ and $\langle M_2 \rangle \notin P$.
- Language A is **mapping reducible** (many-one reducible) to language B, written $A \leq_m B$, if there is a computable function $f : \Sigma^* \longrightarrow \Sigma^*$, where for every w,

  $$w \in A \Longleftrightarrow f(w) \in B.$$

- $A \leq_m B$ is equivalent to $\overline{A} \leq_m \overline{B}$.

- $3SAT = \{\langle \phi \rangle \mid \phi$ is a satisfiable 3CNF-formula$\}$. (A 3CNF-formula is a CNF-formula where all the clauses have three literals.) $3SAT$ is NP-complete.

- $VERTEX\_COVER = \{\langle G, k \rangle \mid G$ is an undirected graph that has a k-node vertex cover$\}$. (A *vertex cover* of an undirected graph G is a subset of the nodes where every edge of G touches one of those nodes.) $VERTEX\_COVER$ is NP-complete.

- $SUBSET\_SUM = \{\langle S, t \rangle \mid S = \{x_1, \cdots, x_k\}$ and for some $\{y_1, \cdots, y_l\} \subseteq S$, we have $\sum y_i = t\}$. $SUBSET\_SUM$ is NP-complete.