# Midterm

## Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

## Problems

1. Let $A = \{a, b, c, d, e, f\}$ and $R = \{(a, c), (b, e), (e, f)\}$, which is a binary relation on $A$.

   (a) Give a symmetric and transitive but *not* reflexive binary relation on $A$ that includes $R$. Please present the relation using a directed graph.

   (b) Find the smallest equivalence relation on $A$ that includes $R$. Please present the relation using a directed graph.

2. Let $L$ be a language over $\Sigma$ (i.e., $L \subseteq \Sigma^*$). Two strings $x$ and $y$ in $\Sigma^*$ are *distinguishable by $L$* if for some string $z$ in $\Sigma^*$, exactly one of $xz$ and $yz$ is in $L$. When no such $z$ exists, i.e., for every $z$ in $\Sigma^*$, either both of $xz$ and $yz$ or neither of them are in $L$, we say that $x$ and $y$ are *indistinguishable by $L$*. Is indistinguishability by a language an equivalence relation (over $\Sigma^*$)? Please justify your answer.

3. Give the state diagrams of DFAs recognizing the following languages. In all parts, the alphabet is $\{\mathtt{0}, \mathtt{1}\}$.

   (a) $\{w \mid w \text{ contains the substring } \mathtt{1010}, \text{ i.e., } w = x\mathtt{1010}y \text{ for some } x \text{ and } y\}$.

   (b) $\{w \mid \text{every odd position of } w \text{ is a } \mathtt{0}\}$ (Note: see $w$ as $w_1 w_2 \cdots w_n$, where $w_i \in \{\mathtt{0}, \mathtt{1}\}$).

4. Let $L = \{w \in \{0, 1\}^* \mid w \text{ contains 101 as a substring or ends with a 1}\}$.

   (a) Draw the state diagram of an NFA, with as few states as possible, that recognizes $L$. The fewer states your NFA has, the more points you will be credited for this problem.

   (b) Convert the preceding NFA systematically into an equivalent DFA (using the procedure discussed in class). Do not attempt to optimize the number of states, though you may omit the unreachable states.

5. Give the state diagram of a DFA that recognizes the following language:

$$C_6 = \{x \mid x \text{ is a binary number that is a multiple of } 6\}.$$

6. Consider the following CFG discussed in class, where for convenience the variables have been renamed with single letters.

$$\begin{array}{lcl} E & \rightarrow & E+T \mid T \\ T & \rightarrow & T \times F \mid F \\ F & \rightarrow & (E) \mid a \end{array}$$

   Give the (leftmost) derivation and parse tree for the string $(a+(a)) \times a$.

7. Give a context-free grammar that generates the following language: $\{w \in \{a,b,c\}^* \mid$ the number of $a$'s in $w$ equals that of $b$'s or $c$'s$\}$ (no restriction is imposed on the order in which the input symbols may appear). Please make the CFG as simple as possible and explain the intuition behind it.

8. For two given languages $A$ and $B$, define $A \diamond B = \{xy \mid x \in A \text{ and } y \in B \text{ and } |x| = |y|\}$. Prove that, if $A$ and $B$ are regular, then $A \diamond B$ is context-free. (Hint: construct a PDA where the stack is used to ensure that $x$ and $y$ are of equal length.)

9. Prove, using the pumping lemma, that $\{1^{n^2} \mid n \geq 0\}$ is not context free.

10. For languages $A$ and $B$, let the *perfect shuffle* of $A$ and $B$ be the language $\{w \mid w = a_1 b_1 \cdots a_k b_k$, where $a_1 \cdots a_k \in A$ and $b_1 \cdots b_k \in B$, each $a_i, b_i \in \Sigma\}$. Show that the class of context-free languages is *not* closed under perfect shuffle.

## Appendix

- Common properties of a binary relation $R$ on $A$:
  - $R$ is *reflexive* if for every $x \in A$, $xRx$.
  - $R$ is *symmetric* if for every $x, y \in A$, $xRy$ if and only if $yRx$.
  - $R$ is *transitive* if for every $x, y, z \in A$, $xRy$ and $yRz$ implies $xRz$.
- (Pumping Lemma for Context-Free Languages) If $A$ is a context-free language, then there is a number $p$ such that, if $s$ is a string in $A$ and $|s| \geq p$, then $s$ may be divided into five pieces, $s = uvxyz$, satisfying the conditions: (1) for each $i \geq 0$, $uv^ixy^iz \in A$, (2) $|vy| > 0$, and (3) $|vxy| \leq p$.