Final

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. Consider the following CFG discussed in class, where for convenience the variables have been renamed with single letters.

$$
\begin{array}{lcl}
E & \rightarrow & E + T \mid T \\
T & \rightarrow & T \times F \mid F \\
F & \rightarrow & (E) \mid a
\end{array}
$$

   Convert the grammar into an equivalent PDA (that recognizes the same language). Please use the standard form for a transition; no shorthands.

2. Show that (single-tape) Turing machines which are allowed to move its head only to the right are less powerful than the usual Turing machines. What class of languages do this type of restricted Turing machines recognize? Please sketch a proof.

3. A *useless state* in a pushdown automaton is a state that is never entered on any input. Show the decidability of the problem of determining whether a given pushdown automaton has a useless state.

4. Let A and B be two disjoint languages. Say that language C *separates* A and B if $A \subseteq C$ and $B \subseteq \overline{C}$. Show that any two disjoint co-Turing-recognizable languages are separable by some decidable language.

5. Discuss the applicability of Rice's Theorem for each of the following problems (languages). Please give the reasons why or why not.

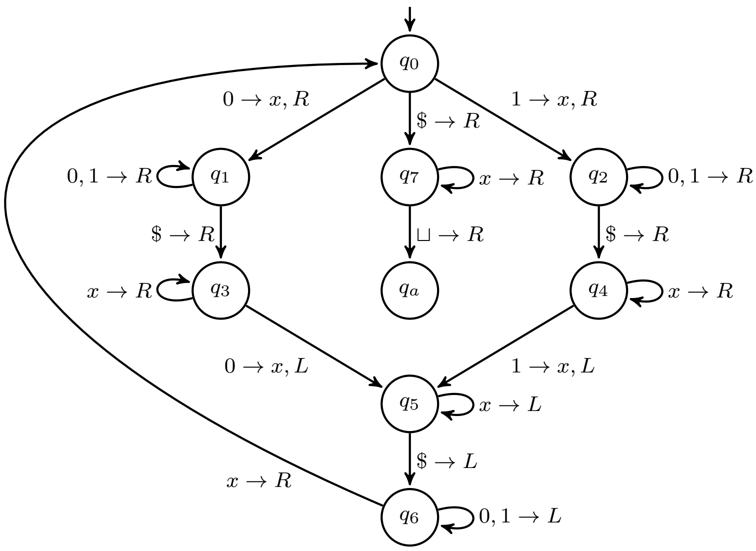
   (a) $REGULAR_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is regular}\}$.

   (b) $COUNTABLE_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is countable}\}$.

6. Show that if A is Turing-recognizable and $A \leq_m \overline{A}$, then A is decidable.

7. Prove that $HALT_{\text{TM}} \leq_m \overline{E_{\text{TM}}}$, where $HALT_{\text{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on } w\}$ and $E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$. From this, what can you say about Turing reducibility and mapping reducibility?

8. In the proof of the Cook-Levin theorem, which states that SAT is NP-complete, we used 2×3 windows of cells to formulate the constraint that the configuration of each row (except the first one) in the $n^k \times n^k$ tableau legally follows the configuration of the preceding row. Suppose the Turing machine being reduced is for the NP language $\{W\$W \mid W \in \{0,1\}^*\}$, as shown below.



Which of the following 2×3 windows of cells are illegal? Why?

\$	0	1
q_2	0	1

1	1	\$
1	1	q_6

1	q_0	0
1	x	q_2

q_4	1	0
1	q_5	0

x	x	q_7
x	x	q_a

1	0	q_0
1	0	\$

9. In the proof that the *3SAT* problem is polynomially reducible to the *CLIQUE* problem, we convert an arbitrary boolean expression in 3CNF (input of the *3SAT* problem) to an input graph of the *CLIQUE* problem.

   (a) Please illustrate the conversion by drawing the graph that will be obtained from the following boolean expression:

   $$(\overline{x} + y + z) \cdot (w + \overline{y} + \overline{z}) \cdot (\overline{w} + x + \overline{y}).$$

   (b) The original boolean expression is satisfiable. As a demonstration of why the reduction is correct, please use the resulting graph to argue that it is indeed the case.

10. Let $DOUBLE\_SAT = \{\langle \phi \rangle \mid \phi \text{ is a Boolean formula with at least two satisfying assignments}\}$. Prove that $DOUBLE\_SAT$ is NP-complete.

Appendix

- $E_{\text{CFG}} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$. E_{CFG} is a decidable language.
- A language is *co-Turing-recognizable* if it is the complement of a Turing-recognizable language.

- (**Rice's Theorem**) Any problem P about Turing machines satisfying the following two conditions is undecidable:

  1. For any TMs M_1 and M_2, where $L(M_1) = L(M_2)$, we have $\langle M_1 \rangle \in P$ iff $\langle M_2 \rangle \in P$.
  2. P is nontrivial, i.e., there exist TMs M_1 and M_2 such that $\langle M_1 \rangle \in P$ and $\langle M_2 \rangle \notin P$.

- Language A is **mapping reducible** (many-one reducible) to language B, written $A \leq_m B$, if there is a computable function $f : \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \Longleftrightarrow f(w) \in B.$$

- $A \leq_m B$ is equivalent to $\overline{A} \leq_m \overline{B}$.

- $SAT = \{\langle \phi \rangle \mid \phi \text{ is a satisfiable Boolean formula}\}$. SAT is NP-complete (the Cook-Levin theorem).

- $3SAT = \{\langle \phi \rangle \mid \phi \text{ is a satisfiable 3CNF-formula}\}$. (A 3CNF-formula is a CNF-formula where all the clauses have three literals.) $3SAT$ is NP-complete.

- $CLIQUE = \{\langle G, k \rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$. (A *clique* in an undirected graph is a subgraph, wherein every two nodes are connected by an edge, and a *k-clique* is a clique that contains k nodes.) $CLIQUE$ is NP-complete.