Final

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. For two given languages A and B, define $A \diamond B = \{xy \mid x \in A \text{ and } y \in B \text{ and } |x| = |y|\}$. Prove that, if A and B are regular, then $A \diamond B$ is context-free. (Hint: construct a PDA where the stack is used to ensure that x and y are of equal length.)

2. Give the implementation-level description of a (single-tape deterministic) Turing machine that decides the language $\{1^n\#1^{n^2} \mid n \geq 1\}$.

3. Show that Turing machines with a *doubly infinite* tape are not more powerful than those with a tape that is infinite only on one end.

4. Show that (single-tape) Turing machines which are allowed to move its head only to the right are less powerful than the usual Turing machines. What class of languages do this type of restricted Turing machines recognize? Please sketch a proof.

5. A *useless state* in a pushdown automaton is a state that is never entered on any input. Show the decidability of the problem of determining if a given pushdown automaton has a useless state.

6. Prove that there exists an undecidable subset of $\{1\}^*$.

7. Prove that $HALT_{\text{TM}} \leq_m E_{\text{TM}}$, where $HALT_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on } w\}$ and $E_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$.

8. Discuss the applicability of Rice's Theorem for each of the following problems (languages). Please give the reasons why or why not.

   (a) $REGULAR_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ is regular}\}$.

   (b) $SMALL50_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM that has no more than 50 states}\}$.

   (c) $FINITE_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ is finite}\}$.

(d) $\mathit{UNCOUNTABLE}_{\mathrm{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is uncountable}\}$.

9. Show that NP is closed under *union* and *concatenation*. It is unknown if NP is also closed under *complement*. Can you explain why determining this closure property is nontrivial?

10. Show that, if P=NP, then every language in P, except $\emptyset$ and Σ^*, is NP-complete.

Appendix

- Language A is **mapping reducible** (many-one reducible) to language B, written $A \leq_m B$, if there is a computable function $f : \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \Longleftrightarrow f(w) \in B.$$

- $A \leq_m B$ is equivalent to $\overline{A} \leq_m \overline{B}$.

- **Rice's Theorem** states that any problem P about Turing machines satisfying the following two conditions is undecidable:

  1. For any TMs M_1 and M_2, where $L(M_1) = L(M_2)$, we have $\langle M_1 \rangle \in P$ iff $\langle M_2 \rangle \in P$.
  2. P is nontrivial, i.e., there exist TMs M_1 and M_2 such that $\langle M_1 \rangle \in P$ and $\langle M_2 \rangle \notin P$.

- **NP** is the class of *polynomially verifiable* languages, i.e., languages that have polynomial time verifiers.

- A language is in NP iff it is decided by some nondeterministic polynomial time Turing machine.

- Language A is **polynomial time mapping reducible** (polynomial time reducible) to language B, written $A \leq_p B$, if there is a *polynomial time* computable function $f : \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \Longleftrightarrow f(w) \in B.$$

- A language B is **NP-complete** if it satisfies the following two conditions:

  1. B is in NP, and
  2. every A in NP is polynomial time reducible to B.