Midterm

(May 2, 2002)

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. Let R_1 and R_2 be binary relations on a set A, i.e., $R_1, R_2 \subseteq A \times A$. Prove that, if R_1 and R_2 are equivalence relations, then $R_1 \cap R_2$ (the intersection of R_1 and R_2) is also an equivalence relation on A.

2. (a) Draw the state diagram of an NFA, with as few states as possible, that recognizes $\{w \in \{0,1\}^* \mid w \text{ ends with } 0 \text{ or } 01\}$.

   (b) Convert the NFA in (a) systematically into an equivalent DFA (using the procedure discussed in class); do not attempt to optimize the number of states.

3. (a) Draw the state diagram of a DFA (with as few states as possible) that recognizes $\{w \in \{0,1\}^* \mid w \text{ doesn't contain } 000 \text{ or } 111 \text{ as a substring}\}$.

   (b) Translate the DFA in (a) systematically to an equivalent context-free grammar (using the procedure discussed in class).

4. Write a regular expression for the language in Problem 3.

5. Let $A = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and if } i = 2 \text{ then } j < k\}$. Show that A satisfies the pumping lemma for regular languages. What is the (smallest) pumping length of A?

6. Show that, if G is a CFG in Chomsky normal form, then any string $w \in L(G)$ of length $n \geq 1$, exactly $2n - 1$ steps are required for any derivation of w.

7. Draw the state diagram of a pushdown automaton (PDA) that recognizes the following language: $\{w\#x \mid w^{\mathcal{R}} \text{ is a substring of } x \text{ for } w, x \in \{0,1\}^*\}$. Please explain the intuition behind the PDA.

8. Prove that the class of context-free languages is not closed under either *intersection* or *complement*. (Hint: the class of context-free languages is known to be closed under union.)

9. We have shown in class that $\{1^{n^2} \mid n \geq 0\}$ is not regular. Is it context-free? Prove your answer.

10. Find a regular language A, a non-regular but context-free language B, and a non-context-free language C over $\{0,1\}$ such that $C \subseteq B \subseteq A$.

Appendix

- A context-free grammar is in **Chomsky normal form** if every rule is of the form

$$\begin{array}{lcl} A & \rightarrow & BC \quad \text{or} \\ A & \rightarrow & a \end{array}$$

  where a is any terminal and A, B, and C are any variables—except that B and C may not be the start variable. In addition,

$$S \rightarrow \varepsilon$$

  is permitted if S is the start variable.

- If A is a regular language, then there is a number p (the pumping length) such that, if s is any string in A and $|s| \geq p$, then s may be divided into three pieces, $s = xyz$, satisfying the conditions: (1) for each $i \geq 0$, $xy^iz \in A$, (2) $|y| > 0$, and (3) $|xy| \leq p$.

- If A is a context-free language, then there is a number p such that, if s is a string in A and $|s| \geq p$, then s may be divided into five pieces, $s = uvxyz$, satisfying the conditions: (1) for each $i \geq 0$, $uv^ixy^iz \in A$, (2) $|vy| > 0$, and (3) $|vxy| \leq p$.