Midterm

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. Give the state diagrams of DFAs recognizing the following languages. In all parts, the alphabet is $\{\mathtt{0}, \mathtt{1}\}$.
   (a) $\{w \mid w$ contains the substring $\mathtt{1001}$, i.e., $w = x\mathtt{1001}y$ for some x and $y\}$.
   (b) $\{w \mid$ every even position of w is a $\mathtt{0}\}$ (Note: see w as $w_1 w_2 \cdots w_n$, where $w_i \in \{\mathtt{0}, \mathtt{1}\}$).
2. Let $L = \{w \in \{0, 1\}^* \mid w$ contains 100 as a substring or ends with a $0\}$.
   (a) Draw the state diagram of an NFA, with as few states as possible, that recognizes L. The fewer states your NFA has, the more points you will be credited for this problem.
   (b) Convert the preceding NFA systematically into an equivalent DFA (using the procedure discussed in class). Do not attempt to optimize the number of states, though you may omit the unreachable states.
3. Let $L = \{1^p \mid p$ is a prime number less than $2^{2^{10}}\}$. Is L a regular language? Why or why not?
4. Give the state diagram of a DFA that recognizes the following language:
   $$C_5 = \{x \mid x \text{ is a binary number that is a multiple of } 5\}.$$
5. Given a language $L \subseteq \Sigma^*$, an equivalence relation R_L over Σ^* is defined follows:
   $$xR_L y \quad \text{iff} \quad \forall z \in \Sigma^*(xz \in L \leftrightarrow yz \in L)$$
   Suppose $L = \{w \mid w$ contains the substring $\mathtt{101}$, i.e., $w = x\mathtt{101}y$ for some x and $y\}$. What are the equivalence classes determined by R_L? Please give an intuitive verbal description for each of the equivalence classes.
6. Consider the following CFG discussed in class, where for convenience the variables have been renamed with single letters.
   $$\begin{array}{lcl} E & \rightarrow & E + T \mid T \\ T & \rightarrow & T \times F \mid F \\ F & \rightarrow & (E) \mid a \end{array}$$
   Give the (leftmost) derivation and parse tree for the string $a \times ((a) + a)$.

7. Prove that, if C is a context-free language and R a regular language, then $C \cap R$ is context-free. (Hint: combine the finite control part of a PDA and that of an NFA.)

8. For two given languages A and B, define $A \diamond B = \{xy \mid x \in A \text{ and } y \in B \text{ and } |x| = |y|\}$. Prove that, if A and B are regular, then $A \diamond B$ is context-free. (Hint: construct a PDA where the stack is used to ensure that x and y are of equal length.)

9. Prove, using the pumping lemma, that $\{a^m b^n c^{m+n} \mid m, n \geq 1\}$ is not context free.

10. For languages A and B, let the *perfect shuffle* of A and B be the language $\{w \mid w = a_1 b_1 \cdots a_k b_k$, where $a_1 \cdots a_k \in A$ and $b_1 \cdots b_k \in B$, each $a_i, b_i \in \Sigma\}$. Show that the class of context-free languages is *not* closed under perfect shuffle.

Appendix

- (Pumping Lemma for Context-Free Languages) If A is a context-free language, then there is a number p such that, if s is a string in A and $|s| \geq p$, then s may be divided into five pieces, $s = uvxyz$, satisfying the conditions: (1) for each $i \geq 0$, $uv^ixy^iz \in A$, (2) $|vy| > 0$, and (3) $|vxy| \leq p$.