Final

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. Let $C_5 = \{x \mid x \text{ is a binary number that is a multiple of } 5\}$. Show that C_5 is regular.

2. Draw the state diagram of a pushdown automaton (PDA) that recognizes the following language: $\{w \in \{a, b, c\}^* \mid \text{ the number of } a\text{'s in } w \text{ equals that of } b\text{'s or } c\text{'s}\}$. Please make the PDA as simple as possible and explain the intuition behind the PDA.

3. Give the implementation-level description of a (single-tape deterministic) Turing machine that decides the language $\{1^n\#1^{2^n} \mid n \geq 1\}$.

4. Let $EQ_{\text{CFG}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$. Show that EQ_{CFG} is undecidable.

5. Define a *two-headed finite automaton* (2DFA) to be a deterministic finite automaton that has two read-only, bidirectional heads that start at the left-hand end of the input tape and can be independently controlled to move in either direction. The tape of a 2DFA is finite and is just large enough to contain the input plus two additional blank tape cells, one on the left-end and one on the right-hand end, that serve as delimiters. A 2DFA accepts its input by entering a special accept state. For example, a 2DFA can recognize the language $\{a^nb^nc^n \mid n \geq 0\}$.

   Let $E_{\text{2DFA}} = \{\langle M\rangle \mid M \text{ is a 2DFA and } L(M) = \emptyset\}$. Show that E_{2DFA} is undecidable.

6. Discuss the applicability of Rice's Theorem for each of the following problems (languages). Please give the reasons why or why not.

   (a) $\mathit{REGULAR}_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ is regular}\}$.

   (b) $\mathit{UNCOUNTABLE}_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ is uncountable}\}$.

7. Prove that $\mathit{HALT}_{\text{TM}} \leq_m E_{\text{TM}}$, where $\mathit{HALT}_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on } w\}$ and $E_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$.

8. In the proof of the Cook-Levin theorem, which states that SAT is NP-complete, we used 2×3 windows of cells to formulate the constraint that the configuration of each row (except the first one) in the $n^k \times n^k$ tableau legally follows the configuration of

the preceding row. Why couldn't we simply use two entire rows of cells to formulate the constraint?

9. In the proof (discussed in class) of the NP-completeness of the *CLIQUE* problem by reduction from the *3SAT* problem, we convert an arbitrary boolean expression in 3CNF (input of the *3SAT* problem) to an input graph of the *CLIQUE* problem.

   (a) Please illustrate the conversion by drawing the graph that will be obtained from the following boolean expression:

   $$(x + y + \overline{z}) \cdot (w + \overline{y} + z) \cdot (\overline{w} + x + y).$$

   (b) The original boolean expression is satisfiable. As a demonstration of how the reduction works, please use the resulting graph to argue that it is indeed the case.

10. Let $DOUBLE\_SAT = \{\langle \phi \rangle \mid \phi \text{ is a Boolean formula with at least two satisfying assignments}\}$. Prove that $DOUBLE\_SAT$ is NP-complete. (Hint: reduction from the SAT problem; introduce a fresh (new) variable ...)

Appendix

- $A_{\mathrm{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$. A_{TM} is undecidable.
- $ALL_{\mathrm{CFG}} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \Sigma^*\}$. ALL_{CFG} is undecidable.
- **Rice's Theorem** states that any problem P about Turing machines satisfying the following two conditions is undecidable:
  1. For any TMs M_1 and M_2, where $L(M_1) = L(M_2)$, we have $\langle M_1 \rangle \in P$ iff $\langle M_2 \rangle \in P$.
  2. P is nontrivial, i.e., there exist TMs M_1 and M_2 such that $\langle M_1 \rangle \in P$ and $\langle M_2 \rangle \notin P$.
- Language A is **mapping reducible** (many-one reducible) to language B, written $A \leq_m B$, if there is a computable function $f : \Sigma^* \longrightarrow \Sigma^*$, where for every w,

  $$w \in A \Longleftrightarrow f(w) \in B.$$

- $A \leq_m B$ is equivalent to $\overline{A} \leq_m \overline{B}$.
- $SAT = \{\langle \phi \rangle \mid \phi \text{ is a satisfiable Boolean formula}\}$. SAT is NP-complete (the Cook-Levin theorem).