Midterm

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. Give the state diagrams of DFAs, with as few states as possible, recognizing the following languages.
   (a) $\{w \in \{0,1\}^* \mid w \text{ begins with a 0 and ends with a 1}\}$.
   (b) $\{w \in \{0,1\}^* \mid w \text{ doesn't contain the substring 011}\}$.
2. Let $L = \{w \in \{0,1\}^* \mid w \text{ contains 011 as a substring or ends with a 1}\}$.
   (a) Draw the state diagram of an NFA, with as few states as possible, that recognizes L. The fewer states your NFA has, the more points you will be credited for this problem.
   (b) Convert the preceding NFA systematically into an equivalent DFA (using the procedure discussed in class). Do not attempt to optimize the number of states, though you may omit the unreachable states.
3. Let $L = \{1^p \mid p \text{ is a prime number less than } 2^{2^{10}}\}$. Is L a regular language? Why or why not?
4. For languages A and B, let the *shuffle* of A and B be the language $\{w \mid w = a_1 b_1 \cdots a_k b_k$, where $a_1 \cdots a_k \in A$ and $b_1 \cdots b_k \in B$, each $a_i, b_i \in \Sigma^*\}$. Show that the class of regular languages is closed under shuffle.
5. A *synchronizing sequence* for a DFA $M = (Q, \Sigma, \delta, q_0, F)$ and some "home" state $h \in Q$ is a string $s \in \Sigma^*$ such that, for every $q \in Q$, $\delta(q, s) = h$. A DFA is said to be *synchronizable* if it has a synchronizing sequence for some state. Prove that, if M is a k-state synchronizable DFA, then it has a synchronizing sequence of length at most k^3. (Note: $\delta(q, s)$ equals the state where M ends up when M starts from state q and reads input s.)
6. Consider the following CFG discussed in class, where for convenience the variables have been renamed with single letters.

$$\begin{array}{lcl} E & \rightarrow & E + T \mid T \\ T & \rightarrow & T \times F \mid F \\ F & \rightarrow & (E) \mid a \end{array}$$

   Give the (leftmost) derivation and parse tree for the string $(a + (a)) \times a$.

7. Draw the state diagram of a pushdown automaton (PDA) that recognizes the following language: $\{w \in \{a, b, c\}^* \mid$ the number of a's in w equals that of b's or c's$\}$ (no restriction is imposed on the order in which the symbols may appear). Please make the PDA as simple as possible and explain the intuition behind the PDA.

8. For two given languages A and B, define $A \diamond B = \{xy \mid x \in A \text{ and } y \in B \text{ and } |x| = |y|\}$. Prove that, if A and B are regular, then $A \diamond B$ is context-free. (Hint: construct a PDA where the stack is used to ensure that x and y are of equal length.)

9. Prove that the class of context-free languages is not closed under *complement*.

10. Prove, using the pumping lemma, that $\{a^m b^n c^{m \times n} \mid m, n \geq 1\}$ is not context free.

Appendix

- (Pumping Lemma for Context-Free Languages)

  If A is a context-free language, then there is a number p such that, if s is a string in A and $|s| \geq p$, then s may be divided into five pieces, $s = uvxyz$, satisfying the conditions:

  1. for each $i \geq 0$, $uv^ixy^iz \in A$,
  2. $|vy| > 0$, and
  3. $|vxy| \leq p$.