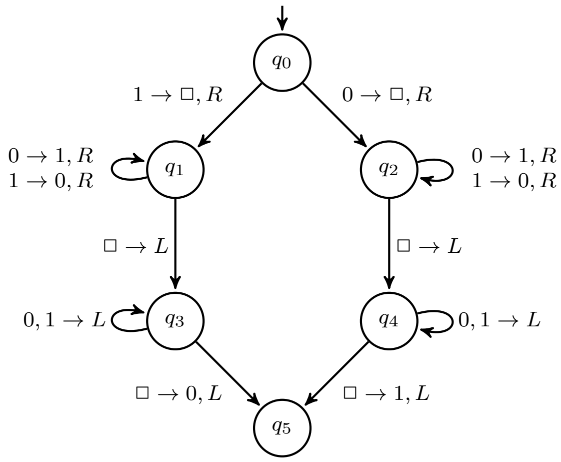
Final

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. Below is a formal description of a Turing machine that computes a function from $\{0,1\}^*$ to $\{0,1\}^*$. Explain in words what exactly the machine computes.



2. Give a formal description of a (single-tape deterministic) Turing machine that decides the language $\{1^k\#1^{2^k} \mid k \geq 1\}$. Also, analyze the time complexity of the Turing machine.

3. Show that single-tape TMs that cannot write on the portion of the tape containing the input string recognize only regular languages.

4. Let A and B be two disjoint languages. Say that language C *separates* A and B if $A \subseteq C$ and $B \subseteq \overline{C}$. Show that any two disjoint co-Turing-recognizable languages are separable by some decidable language.

5. Let $EQ_{\text{CFG}} = \{\langle G, H\rangle \mid G$ and H are CFGs and $L(G) = L(H)\}$. Show that EQ_{CFG} is undecidable.

6. Discuss the applicability of Rice's Theorem for each of the following problems (languages). Please give the reasons why or why not.

   (a) $\{\langle M\rangle \mid M$ is a TM and $10^*1 \subseteq L(M)\}$.

(b) $UNCOUNTABLE_{\mathrm{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) \text{ is uncountable}\}$.

7. Prove that $HALT_{\mathrm{TM}} \le_m \overline{E_{\mathrm{TM}}}$, where $HALT_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M$ halts on $w\}$ and $E_{\mathrm{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$.

8. In the proof of the Cook-Levin theorem, which states that *SAT* is NP-complete, we used 2×3 windows of cells to formulate the constraint that the configuration of each row (except the first one) in the $n^k \times n^k$ tableau legally follows the configuration of the preceding row. Consider the machine in Problem 1. Which of the following 2×3 windows of cells are illegal? Why?

$$\begin{array}{|c|c|c|}\hline q_0 & 1 & 0 \\ \hline \square & q_1 & 0 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & q_2 & 0 \\ \hline 1 & 1 & q_2 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline \square & 0 & 1 \\ \hline \square & 0 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 0 \\ \hline 1 & 1 & q_4 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline q_5 & 0 & 1 \\ \hline q_5 & 0 & 1 \\ \hline\end{array}$$

9. In the proof that the *3SAT* problem is polynomially reducible to the *CLIQUE* problem, we convert an arbitrary boolean expression in 3CNF (input of the *3SAT* problem) to an input graph of the *CLIQUE* problem.

   (a) Please illustrate the conversion by drawing the graph that will be obtained from the following boolean expression:

   $$(x + \overline{y} + z) \cdot (w + \overline{y} + \overline{z}) \cdot (\overline{w} + x + y).$$

   (b) The original boolean expression is satisfiable. As a demonstration of why the reduction is correct, please use the resulting graph to argue that it is indeed the case.

10. Let $DOUBLE\_SAT = \{\langle\phi\rangle \mid \phi$ is a Boolean formula with at least two satisfying assignments$\}$. Prove that *DOUBLE_SAT* is NP-complete. (Hint: reduction from the *SAT* problem; introduce a fresh (new) variable ...)

Appendix

- $A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$. A_{TM} is undecidable.
- $ALL_{\mathrm{CFG}} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \Sigma^*\}$. ALL_{CFG} is undecidable.
- **Rice's Theorem** states that any problem P about Turing machines satisfying the following two conditions is undecidable:
  1. For any TMs M_1 and M_2, where $L(M_1) = L(M_2)$, we have $\langle M_1\rangle \in P$ iff $\langle M_2\rangle \in P$.
  2. P is nontrivial, i.e., there exist TMs M_1 and M_2 such that $\langle M_1\rangle \in P$ and $\langle M_2\rangle \notin P$.

- A language is *co-Turing-recognizable* if it is the complement of a Turing-recognizable language.
- Language A is **mapping reducible** (many-one reducible) to language B, written $A \leq_m B$, if there is a computable function $f : \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \Longleftrightarrow f(w) \in B.$$

- $A \leq_m B$ is equivalent to $\overline{A} \leq_m \overline{B}$.
- $SAT = \{\langle \phi \rangle \mid \phi \text{ is a satisfiable Boolean formula}\}$. SAT is NP-complete (the Cook-Levin theorem).