Midterm

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. Draw the state diagram of a DFA, with as few states as possible, that recognizes the language $\{w \in \{0,1\}^* \mid w \text{ doesn't contain the substring } 100\}$.

2. Let $L = \{w \in \{0,1\}^* \mid w \text{ contains } 100 \text{ as a substring or ends with a } 1\}$.

   (a) Draw the state diagram of an NFA, with as few states as possible, that recognizes L. The fewer states your NFA has, the more points you will be credited for this problem.

   (b) Give a regular expression that describes L. The shorter your regular expression is, the more points you will be credited for this problem.

3. For languages A and B, let the *shuffle* of A and B be the language $\{w \mid w = a_1b_1 \cdots a_kb_k, \text{ where } a_1 \cdots a_k \in A \text{ and } b_1 \cdots b_k \in B, \text{ each } a_i, b_i \in \Sigma^*\}$. Show that the class of regular languages is closed under shuffle.

4. Given a language $L \subseteq \Sigma^*$, an equivalence relation R_L over Σ^* is defined follows:

$$xR_Ly \text{ iff } \forall z \in \Sigma^*(xz \in L \leftrightarrow yz \in L).$$

   Suppose $L = \{w \in \{0,1\}^* \mid w \text{ contains the substring } \texttt{100}\}$. What are the equivalence classes determined by R_L? Please give an intuitive verbal description for each of the equivalence classes.

5. An *all*-NFA M is a 5-tuple $(Q, \Sigma, \delta, q, F)$ that accepts $x \in \Sigma^*$ if every possible state that M could be after reading input x is a state from F. Note, in contrast, that an ordinary NFA accepts a string if *some* state among these possible states is an accept state. Please give a formal definition of this computation model, as we did in class for an NFA, including a formal definition of the computation of an all-NFA on some input word.

6. Consider the following CFG discussed in class, where for convenience the variables have been renamed with single letters.

$$\begin{aligned} E &\to E + T \mid T \\ T &\to T \times F \mid F \\ F &\to (E) \mid a \end{aligned}$$

(a) (10 points) Give the (leftmost) derivation and parse tree for the string $(a \times a) + (a)$.

(b) (10 points) Convert the grammar into an equivalent PDA (that recognize the same language).

7. Draw the state diagram of a pushdown automaton (PDA) that recognizes the following language: $\{w \in \{a, b, c\}^* \mid$ the number of a's in w equals that of b's or c's$\}$ (no restriction is imposed on the order in which the symbols may appear). Please make the PDA as simple as possible and explain the intuition behind the PDA.

8. Prove, using the pumping lemma, that $\{a^m b^n c^{m \times n} \mid m, n \geq 1\}$ is not context free.

9. For languages A and B over Σ, let the *perfect shuffle* of A and B be the language $\{w \mid w = a_1 b_1 \cdots a_k b_k$, where $a_1 \cdots a_k \in A$ and $b_1 \cdots b_k \in B$, each $a_i, b_i \in \Sigma\}$. Show that the class of context-free languages is *not* closed under perfect shuffle.

Appendix

- (Pumping Lemma for Context-Free Languages)

  If A is a context-free language, then there is a number p such that, if s is a string in A and $|s| \geq p$, then s may be divided into five pieces, $s = uvxyz$, satisfying the conditions:

  1. for each $i \geq 0$, $uv^i xy^i z \in A$,
  2. $|vy| > 0$, and
  3. $|vxy| \leq p$.