Midterm

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. (a) Give a symmetric and transitive but not reflexive binary relation on $A = \{a, b, c, d\}$ that includes $\{(a, b), (b, c)\}$; it may be a good idea to represent the relation by a directed graph.

   (b) Let $R = \{(a, c), (b, c), (b, d)\}$ be a binary relation on $A = \{a, b, c, d, e\}$. Find the smallest equivalence relation on A that includes R; again, it may be a good idea to represent the relation by a directed graph.

2. (a) Draw the state diagram of an NFA, with as few states as possible, that recognizes $\{w \in \{0, 1\}^* \mid w \text{ contains 101 as a substring or ends with 1}\}$. The fewer states your NFA has, the more points you will be credited for this problem. (5 points)

   (b) Convert the NFA in (a) systematically into an equivalent DFA (using the procedure discussed in class); do not attempt to optimize the number of states. (10 points)

3. (a) Draw the state diagram of a DFA, with as few states as possible, that recognizes $\{w \in \{0, 1\}^* \mid w \text{ does not contain 101 or 010 as a substring}\}$. The fewer states your DFA has, the more points you will be credited for this problem. (10 points)

   (b) Translate the DFA in (a) systematically to an equivalent context-free grammar (using the procedure discussed in class), which is called regular grammar. (5 points)

4. Write a regular expression for the language in Problem 3.

5. A *synchronizing sequence* for a DFA $M = (Q, \Sigma, \delta, q_0, F)$ and some "home" state $h \in Q$ is a string $s \in \Sigma^*$ such that, for every $q \in Q$, $\delta(q, s) = h$. A DFA is said to be *synchronizable* if it has a synchronizing sequence for some state. Try to find a 4-state synchronizable DFA with a synchronizing sequence as long as possible. The

longer the synchronizing sequence is, the more points you will be credited for this problem.

6. Draw the state diagram of a pushdown automaton (PDA) that recognizes the following language: $\{w \mid w \in \{0,1\}^* \text{ and } w \text{ has more 0's than 1's}\}$. Please make the PDA as simple as possible and explain the intuition behind the PDA.

7. Prove that, if C is a context-free language and R a regular language, then $C \cap R$ is context-free. (Hint: combine the finite control part of a PDA and that of an NFA.)

8. For two given languages A and B, define $A \diamond B = \{xy \mid x \in A \text{ and } y \in B \text{ and } |x| = |y|\}$. Prove that, if A and B are regular, then $A \diamond B$ is context-free. (Hint: construct a PDA where the stack is used to ensure that x and y are of equal length.)

9. Prove, using the pumping lemma, that $\{x\#wxy \mid w, x, y \in \{a,b\}^*\}$ is not context-free. (Hint: consider $s = a^p b^p \# a^p b^p$, where p is the pumping length.)

10. Consider the following context-free grammar:

$$S \quad \rightarrow \quad SS \mid aSaSb \mid aSbSa \mid bSaSa \mid \varepsilon$$

Prove that every string over $\{a,b\}$ with twice as many a's as b's (including the empty string) can be generated from S. (Hint: by induction on the length of a string.) (bonus 10 points)

Appendix

- Properties of a binary relation R on A:
  - R is *reflexive* if for every $x \in A$, xRx.
  - R is *symmetric* if for every $x, y \in A$, xRy if and only if yRx.
  - R is *transitive* if for every $x, y, z \in A$, xRy and yRz implies xRz.
- If A is a context-free language, then there is a number p such that, if s is a string in A and $|s| \geq p$, then s may be divided into five pieces, $s = uvxyz$, satisfying the conditions: (1) for each $i \geq 0$, $uv^ixy^iz \in A$, (2) $|vy| > 0$, and (3) $|vxy| \leq p$.