Final

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. (a) Draw the state diagram of a DFA, with as few states as possible, that recognizes $\{w \in \{0,1\}^* \mid w \text{ doesn't contain 000 or 010 as a substring}\}$. The fewer states your DFA has, the more points you will be credited for this problem.

   (b) Translate the DFA in (a) systematically to an equivalent context-free grammar (using the procedure discussed in class).

2. Give the implementation-level description of a (single-tape deterministic) Turing machine that decides the language $\{1^i\#1^j \mid 0 \leq i \leq j\}$. (15 points)

3. Briefly explain why a pushdown automaton with three stacks are not more powerful (recognizing a larger class of languages) than one with two stacks. (5 points)

4. Prove that a language is decidable if and only if some enumerator enumerates the language in lexicographic order.

5. Prove that EQ_{CFG} is co-Turing-recognizable, where $EQ_{\text{CFG}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$.

6. Show that if A is Turing-recognizable and $A \leq_m \overline{A}$, then A is decidable.

7. Prove that $HALT_{\text{TM}} \leq_m E_{\text{TM}}$, where $HALT_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on } w\}$ and $E_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$.

8. According to Rice's Theorem, any problem P about Turing machines that satisfies the following two properties is undecidable:

   (a) For any TMs M_1 and M_2, where $L(M_1) = L(M_2)$, we have $\langle M_1\rangle \in P$ iff $\langle M_2\rangle \in P$.

(b) There exist TMs M_1 and M_2 such that $\langle M_1 \rangle \in P$ and $\langle M_2 \rangle \notin P$.

Discuss the applicability of Rice's Theorem for each of the following problems (languages). Please give the reasons why or why not.

(a) $CF_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is context-free}\}$.

(b) $SMALL100_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM that has less than 100 states}\}$.

(c) $FINITE_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is finite}\}$.

(d) $COUNTABLE_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is countable}\}$.

9. In the proof of the Cook-Levin theorem, which states that SAT is NP-complete, we used 2×3 windows of cells to formulate the constraint that the configuration of each row (except the first one) in the $n^k \times n^k$ tableau follows legally from the configuration of the preceding row. Why couldn't we use two entire rows of cells directly?

10. What's wrong with the following arguments?

> Consider an algorithm for SAT: "On input ϕ, try all possible assignments to the variables. Accept if any satisfies ϕ." This algorithm clearly requires exponential time. Thus SAT has exponential time complexity. Therefore SAT is not in P. Because SAT is in NP, it must be true that P is not equal to NP.

Appendix

- $A_{\text{CFG}} = \{\langle G, w \rangle \mid G \text{ is a CFG that generates string } w\}$ is decidable.
- A language is *co-Turing-recognizable* if it is the complement of a Turing-recognizable language.
- A language is decidable if and only if it is both Turing-recognizable and co-Turing-recognizable.
- Language A is **mapping reducible** (many-one reducible) to language B, written $A \leq_m B$, if there is a computable function $f : \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \Longleftrightarrow f(w) \in B.$$

- $A \leq_m B$ is equivalent to $\overline{A} \leq_m \overline{B}$.
- $SAT = \{\langle \phi \rangle \mid \phi \text{ is a satisfiable Boolean formula}\}$.