Midterm

Note

This is a closed-book exam. Each problem accounts for 10 points, unless otherwise marked.

Problems

1. Let L be a language over Σ (i.e., $L \subseteq \Sigma^*$). Two strings x and y in Σ^* are *distinguishable by* L if, for some string z in Σ^*, exactly one of xz and yz is in L. When no such z exists, i.e., for every z in Σ^*, either both of xz and yz or neither of them are in L, we say that x and y are *indistinguishable by* L. Is indistinguishability by a language an equivalence relation (over Σ^*)? Please justify your answer.

2. Give the state diagrams of DFAs, with as few states as possible, recognizing the following languages.
   (a) $\{w \in \{0,1\}^* \mid w \text{ begins with a 1 and also ends with a 1}\}$.
   (b) $\{w \in \{0,1\}^* \mid w \text{ doesn't contain the substring 101}\}$.

3. Let $L = \{w \in \{0,1\}^* \mid w \text{ contains 101 as a substring or ends with a 1}\}$.
   (a) Draw the state diagram of an NFA, with as few states as possible, that recognizes L. The fewer states your NFA has, the more points you will be credited for this problem.
   (b) Give a regular expression that describes L. The shorter your regular expression is, the more points you will be credited for this problem.

4. For languages A and B, let the *shuffle* of A and B be the language $\{w \mid w = a_1 b_1 \cdots a_k b_k$, where $a_1 \cdots a_k \in A$ and $b_1 \cdots b_k \in B$, each $a_i, b_i \in \Sigma^*\}$. Show that the class of regular languages is closed under shuffle.

5. Consider the following CFG discussed in class, where for convenience the variables have been renamed with single letters.

$$
\begin{aligned}
E &\rightarrow E + T \mid T \\
T &\rightarrow T \times F \mid F \\
F &\rightarrow (E) \mid a
\end{aligned}
$$

   (a) (10 points) Give the (leftmost) derivation and parse tree for the string $(a + a) \times (a)$.
   (b) (10 points) Convert the grammar into an equivalent PDA (that recognize the same language).

6. Draw the state diagram of a PDA that recognizes the following language: $\{w \in \{0,1\}^* \mid w \text{ has twice as many 1s as 0s}\}$. Please make the PDA as simple and deterministic as possible and explain the intuition behind the PDA.

7. Prove each of the following statements:

   (a) (2 points) The class of context-free languages is closed under *union*.

   (b) (4 points) The class of context-free languages is not closed under *intersection*.

   (c) (4 points) The class of context-free languages is not closed under *complement*.

8. Let A be the language of all palindromes over $\{0,1\}$ with equal numbers of 0s and 1s. Prove, using the pumping lemma, that A is not context free. (Note: a *palindrome* is a string that reads the same forward and backward.)

9. Find a regular language A, a non-regular but context-free language B, and a non-context-free language C over $\{0,1\}$ such that $C \subseteq B \subseteq A$.

Appendix

- (Pumping Lemma for Context-Free Languages)

  If A is a context-free language, then there is a number p such that, if s is a string in A and $|s| \geq p$, then s may be divided into five pieces, $s = uvxyz$, satisfying the conditions:

  1. for each $i \geq 0$, $uv^ixy^iz \in A$,
  2. $|vy| > 0$, and
  3. $|vxy| \leq p$.