Theory of Computing 2020: Decidability

(Based on [Sipser 2006, 2013])

Yih-Kuen Tsay

1 Introduction

Decidability/Solvability

- We shall demonstrate certain problems that can be solved algorithmically and others that cannot.
- Our objective is to explore the limits of algorithmic solvability.
- Why should you study unsolvability?
  - Knowing when a problem is algorithmically unsolvable is useful because then you realize that the problem must be simplified or altered before you can find an algorithmic solution.
  - A glimpse of the unsolvable can stimulate your imagination and help you gain an important perspective on computation.

2 Decidable Languages

Decidable Languages/Problems

- $A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts } w\}$.
- This is the *acceptance problem* (membership problem) for DFAs formulated as a language.

Theorem 1 (4.1)**.** *A_{DFA} is a decidable language.*

- M = "On input $\langle B, w\rangle$, where B is a DFA and w is a string:
  1. Simulate B on input w.
  2. If the simulation ends in an accept state, *accept*; otherwise, reject."

Decidable Languages/Problems (cont.)

- $A_{\text{NFA}} = \{\langle B, w\rangle \mid B \text{ is an NFA that accepts } w\}$.

Theorem 2 (4.2)**.** *A_{NFA} is a decidable language.*

- N = "On input $\langle B, w\rangle$, where B is an NFA and w is a string:
  1. Convert NFA B to an equivalent DFA C.
  2. Run TM M for deciding A_{DFA} (as a "procedure") on input $\langle C, w\rangle$.
  3. If M accepts, *accept*; otherwise, reject."

Decidable Languages/Problems (cont.)

- $A_{\text{REX}} = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates } w\}$.

Theorem 3 (4.3)**.** *A_{REX} is a decidable language.*

- P = "On input $\langle R, w\rangle$, where R is a regular expression and w is a string:

  1. Convert regular expression R to an equivalent DFA A.
  2. Run TM M for deciding A_{DFA} on input $\langle A, w\rangle$.
  3. If M accepts, *accept*; otherwise, reject."

Decidable Languages/Problems (cont.)

- $E_{\text{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$.

Theorem 4 (4.4)**.** *E_{DFA} is a decidable language.*

- T = "On input $\langle A\rangle$, where A is a DFA:

  1. Mark the start state of A.
  2. Repeat Step 3 until no new states get marked.
  3. Mark any state that has a transition coming into it from any state that is already marked.
  4. If no accept state is marked, *accept*; otherwise, reject."

Decidable Languages/Problems (cont.)

- $EQ_{\text{DFA}} = \{\ \langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\ \}$.

Theorem 5 (4.5)**.** *EQ_{DFA} is a decidable language.*

- F = "On input $\langle A, B\rangle$, where A and B are DFAs:

  1. Construct DFA $C = (A \cap \overline{B}) \cup (\overline{A} \cap B)$.
  2. Run TM T for deciding E_{DFA} on input $\langle C\rangle$.
  3. If T accepts, *accept*; otherwise, reject."

Decidable Languages/Problems (cont.)

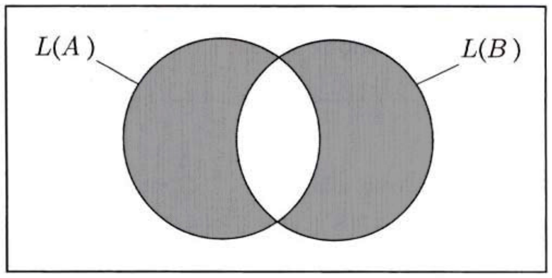


FIGURE 4.6
The symmetric difference of $L(A)$ and $L(B)$

Source: [Sipser 2006]

Decidable CFL Properties

- $A_{\text{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates } w\}$.

Theorem 6 (4.7)**.** *A_{CFG} is a decidable language.*

- $S =$ "On input $\langle G, w\rangle$, where G is a CFG and w is a string:
  1. Convert G to an equivalent grammar in Chomsky normal form.
  2. List all derivations with $2|w| - 1$ steps.
  3. If any of these derivations generate w, *accept*; otherwise, reject."

Decidable CFL Properties (cont.)

- $E_{\text{CFG}} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$.

Theorem 7 (4.8)**.** *E_{CFG} is a decidable language.*

- $R =$ "On input $\langle G\rangle$, where G is a CFG:
  1. Mark all terminals in G.
  2. Repeat Step 3 until no new variables get marked.
  3. Mark any variable A where $A \to U_1U_2\cdots U_k$ is a rule in G and each symbol $U_1, U_2, \cdots, U_k$ has already been marked.
  4. If the start symbol is not marked, *accept*; otherwise, reject."

Decidability of CFLs

Theorem 8 (4.9)**.** *Every context-free language is decidable.*

- Let G be a CFG for the given language A and design a TM M_G that decides A.
- $M_G =$ "On input w:
  1. Run TM S for deciding A_{CFG} on input $\langle G, w\rangle$.
  2. If S accepts, *accept*; otherwise, reject."

Classes of Languages

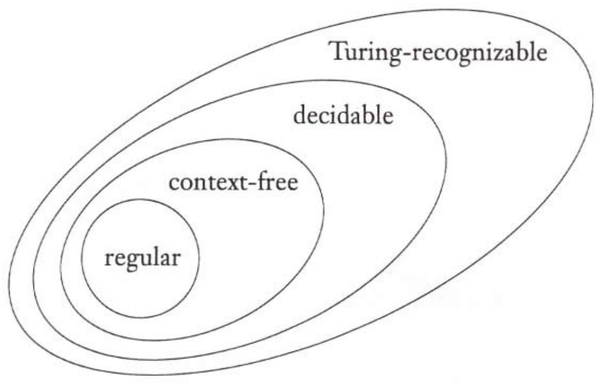


FIGURE 4.10
The relationship among classes of languages

Source: [Sipser 2006]

Classes of Languages (cont.)

Chomsky Hierarchy	Grammar	Language	Computation Model
Type-0	Unrestricted	R.E.	Turing Machine
N/A	(no common name)	Recursive	Decider
Type-1	Context-Sensitive	Context-Sensitive	Linear Bounded
Type-2	Context-Free	Context-Free	Pushdown
Type-3	Regular	Regular	Finite

- Recall that Recursively Enumerable (R.E.) ≡ Turing-recognizable and Recursive ≡ Decidable (Turing-decidable).
- Linear Bounded Automata will be introduced later.

3 The Halting Problem

Undecidability

- We shall prove that *there is a specific problem that is algorithmically unsolvable*.
- This result demonstrates that computers are limited in a very fundamental way.
- Unsolvable problems are not necessarily esoteric. Some ordinary problems that people want to solve may turn out to be unsolvable.
- For example, the general problem of software verification is not solvable by computer.
- The specific problem that we will prove algorithmically unsolvable is the one of *testing whether a Turing machine accepts a given input string*.

The Acceptance Problem

- $A_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$.

Theorem 9 (4.11)**.** *A_{TM} is undecidable.*

- We will prove this fundamental result later.
- On the other hand, A_{TM} is Turing-recognizable.

The Acceptance Problem (cont.)

- $U =$ "On input $\langle M, w\rangle$, where M is a TM and w is a string:
  1. Simulate M on input w.
  2. If M ever enters its accept state, *accept*; if M ever enters its reject state, reject."
- If we had (actually not) some way to determine that M was not *halting* on w, then we could turn the recognizer U into a decider.

Note: The Turing machine U is an example of the *universal Turing machine*, as it is capable of simulating any other Turing machine from the description of that machin. The universal Turing machine inspired "stored-program" computers.

Countable vs. Uncountable Sets

Definition 10 (4.12)**.** Let f be a function from A to B.

- We say that f is *one-to-one* if $f(a) \neq f(b)$ whenever $a \neq b$.
- Say that f is *onto* if, for every $b \in B$, there is an $a \in A$ such that $f(a) = b$.
- A function that is both one-to-one and onto is called a *correspondence*.
- Two sets are considered to have the same size if there is a correspondence between them.

Definition 11 (4.14)**.** A set A is **countable** if either it is finite or it has the same size as $\mathcal{N} = \{1, 2, 3, \cdots\}$; it is **uncountable**, otherwise.

Countable vs. Uncountable Sets (cont.)

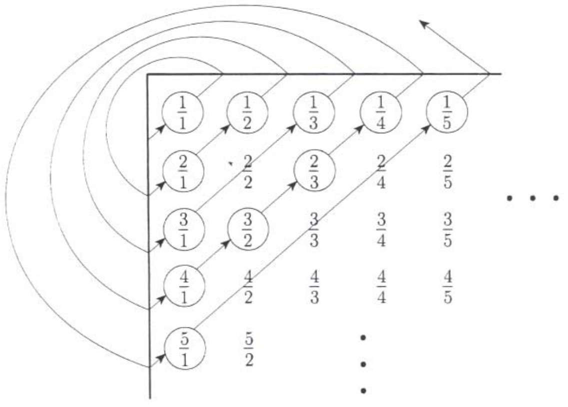


FIGURE **4.16**
A correspondence of $\mathcal{N}$ and $\mathcal{Q}$

Source: [Sipser 2006]

Uncountable Sets

- A real number is one that has a (possibly infinite) decimal representation.
- Let $\mathcal{R}$ be the set of real numbers.

Theorem 12 (4.17)**.** *$\mathcal{R}$ is uncountable.*

Uncountable Sets (cont.)

- Assume that a correspondence f existed between $\mathcal{N}$ and $\mathcal{R}$.

n	$f(n)$
1	$3.\underline{1}4159\cdots$
2	$55.5\underline{5}555\cdots$
3	$0.12\underline{3}45\cdots$
4	$0.500\underline{0}0\cdots$
$\vdots$	$\vdots$

- We can find an x, $0 < x < 1$, so that the i-th digit following the decimal point of x is different from that of $f(i)$; for example, $x = 0.4641\cdots$ is a possible choice.
- This proof technique is called *diagonalization*, discovered by Georg Cantor in 1873.

Unrecognizability

Corollary 13 (4.18)**.** *Some languages are not Turing-recognizable.*

- The set of all Turing machines is countable because each Turing machine M has an encoding into a string $\langle M \rangle$.
- Let $\mathcal{L}$ be the set of all languages over alphabet Σ.
- We can show that there is a correspondence between $\mathcal{L}$ and the uncountable set $\mathcal{B}$ of all infinite binary sequences.
  - Let $\Sigma^* = \{s_1, s_2, s_3, \cdots\}$.
  - Each language $A \in \mathcal{L}$ has a unique sequence in $\mathcal{B}$, where the i-th bit is a 1 if and only if $s_i \in A$.

Undecidability of the Acceptance Problem

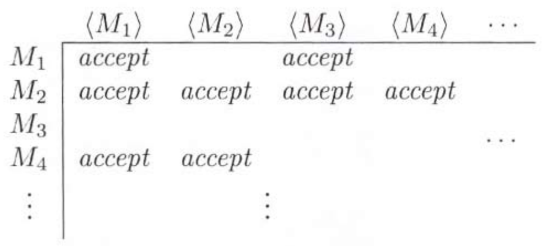
- Suppose H is a decider for A_{TM}:

$$H(\langle M, w \rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ accepts } w \\ \textit{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

- Let D = "On input $\langle M \rangle$, where M is a TM:
  1. Run H on input $\langle M, \langle M \rangle \rangle$.
  2. If H accepts, reject and if H rejects, *accept*."
- When D takes itself, namely $\langle D \rangle$, as input:

$$D(\langle D \rangle) = \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D \rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D \rangle \end{cases}$$

Undecidability of the Acceptance Problem (cont.)

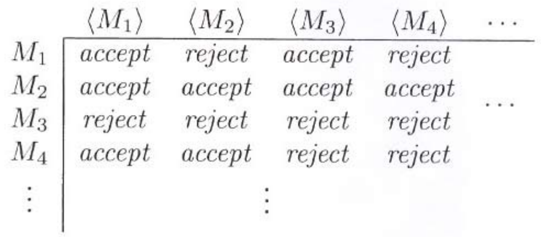


	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	*accept*		*accept*		
M_2	*accept*	*accept*	*accept*	*accept*	
M_3					$\cdots$
M_4	*accept*	*accept*			
$\vdots$			$\vdots$		

FIGURE 4.19
Entry i, j is *accept* if M_i accepts $\langle M_j \rangle$

Source: [Sipser 2006]

Undecidability of the Acceptance Problem (cont.)

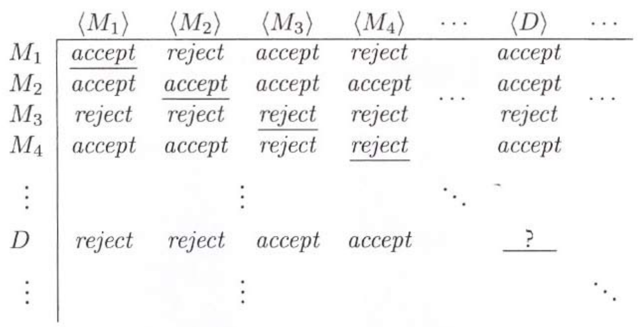


	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*	
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*	
M_4	*accept*	*accept*	*reject*	*reject*	
$\vdots$			$\vdots$		

FIGURE 4.20
Entry i, j is the value of H on input $\langle M_i, \langle M_j\rangle\rangle$

Source: [Sipser 2006]

Undecidability of the Acceptance Problem (cont.)



	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\cdots$	$\langle D\rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*		*accept*	
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*		*reject*	
M_4	*accept*	*accept*	*reject*	*reject*		*accept*	
$\vdots$		$\vdots$			$\ddots$		
D	*reject*	*reject*	*accept*	*accept*		?	
$\vdots$		$\vdots$					$\ddots$

FIGURE 4.21
If D is in the figure, a contradiction occurs at "?"

Source: [Sipser 2006]

A Turing-Unrecognizable Language

- A language is *co-Turing-recognizable* if it is the complement of a Turing-recognizable language.

Theorem 14 (4.22)**.** *A language is decidable if and only if it is both Turing-recognizable and co-Turing-recognizable.*

- Let M_1 be a recognizer for A and M_2 be a recognizer for $\overline{A}$.
- M = "On input w:
  1. Run both M_1 and M_2 on input w in parallel. (M takes turns simulating one step of each machine until one of them halts.)
  2. If M_1 accepts, *accept* and if M_2 accepts, reject."

A Turing-Unrecognizable Language (cont.)

- $\overline{A_{\text{TM}}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ does not accept } w\}$.

Corollary 15 (4.23)**.** $\overline{A_{\text{TM}}}$ *is not Turing-recognizable.*

- A_{TM} is Turing-recognizable, but not decidable.
- From Theorem 4.22, A_{TM} must not be co-Turing-recognizable.
- Therefore, $\overline{A_{\text{TM}}}$ is not Turing-recognizable.