# Theory of Computing 2020: More NP-Complete Problems

(Based on [Sipser 2006, 2013])

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## 1 More NP-Complete Problems

### The Vertex Cover Problem

- A *vertex cover* of an undirected graph $G$ is a subset of the nodes where every edge of $G$ touches one of those nodes.
- $VERTEX\_COVER = \{\langle G, k\rangle \mid G$ is an undirected graph that has a $k$-node vertex cover$\}$.

**Theorem 1.** *$VERTEX\_COVER$ is NP-complete.*

- We show that $3SAT \leq_{\mathrm{P}} VERTEX\_COVER$.

### The Vertex Cover Problem (cont.)

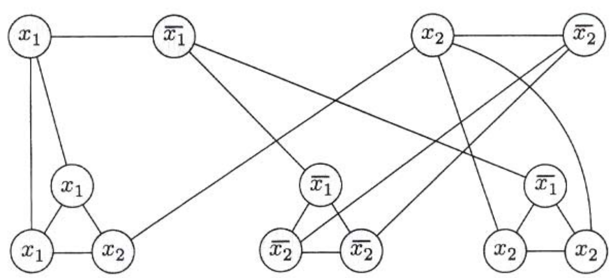


**FIGURE 7.45**
The graph that the reduction produces from
$\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$

Source: [Sipser 2006]

Note: Let $k$ be $m + 2l$, where $m$ is the number of variables and $l$ the number of clauses in $\phi$.

### The Hamiltonian Path Problem

**Theorem 2.** *$HAMPATH$ is NP-complete.*

We show that $3SAT \leq_{\mathrm{P}} HAMPATH$.

### The Hamiltonian Path Problem (cont.)

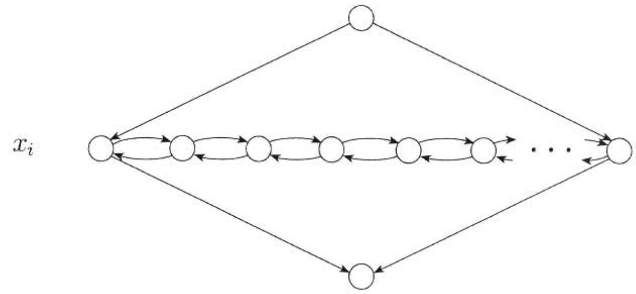


FIGURE 7.47
Representing the variable $x_i$ as a diamond structure

Source: [Sipser 2006]

### The Hamiltonian Path Problem (cont.)


FIGURE 7.48
Representing the clause $c_j$ as a node

Source: [Sipser 2006]

### The Hamiltonian Path Problem (cont.)

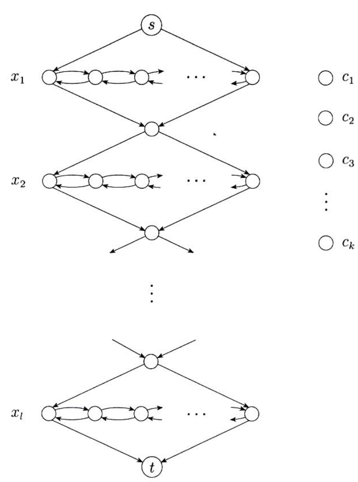


FIGURE 7.49
The high-level structure of $G$

Source: [Sipser 2006]

**The Hamiltonian Path Problem (cont.)**

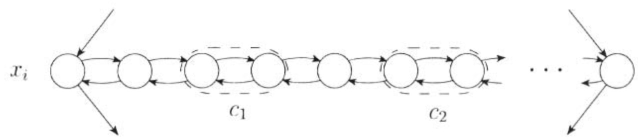


FIGURE **7.50**
The horizontal nodes in a diamond structure

Source: [Sipser 2006]

**The Hamiltonian Path Problem (cont.)**

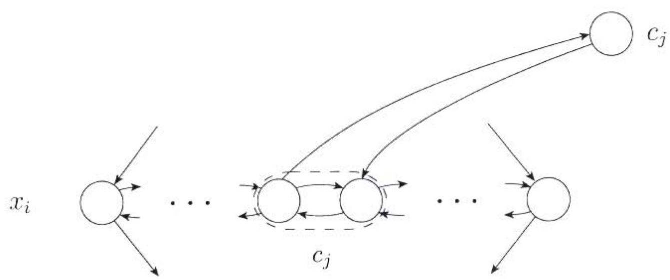


FIGURE **7.51**
The additional edges when clause $c_j$ contains $x_i$

Source: [Sipser 2006]

**The Hamiltonian Path Problem (cont.)**

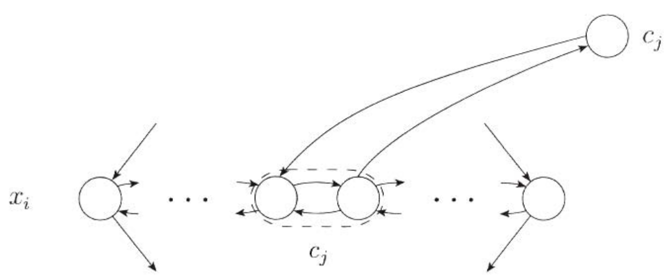


FIGURE **7.52**
The additional edges when clause $c_j$ contains $\overline{x_i}$

Source: [Sipser 2006]

**The Hamiltonian Path Problem (cont.)**

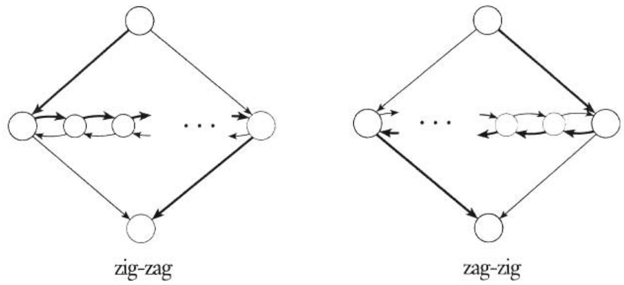


FIGURE 7.53
Zig-zagging and zag-zigging through a diamond, as determined by the satisfying assignment

Source: [Sipser 2006]

### The Hamiltonian Path Problem (cont.)

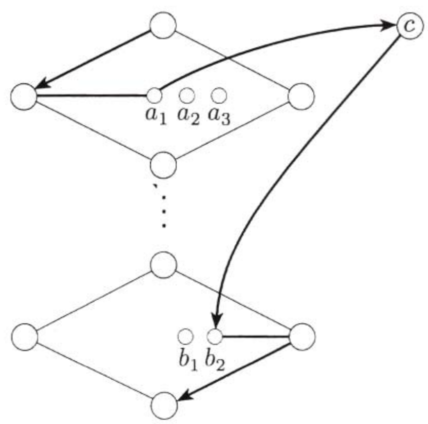


FIGURE 7.54
This situation cannot occur

Source: [Sipser 2006]

### The Hamiltonian Path Problem (cont.)

- Let $UHAMPATH$ be the undirected version of the Hamiltonian path problem $HAMPATH$.

**Theorem 3.** *$UHAMPATH$ is NP-complete.*

- An input $\langle G, s, t\rangle$ for $HAMPATH$ is mapped to $\langle G', s', t'\rangle$ for $UHAMPATH$ as follows.
- Each node $u$ of $G$, except for $s$ and $t$, is replaced by a triple of nodes $u^{\text{in}}$, $u^{\text{mid}}$, and $u^{\text{out}}$ in $G'$.
- Nodes $s$ and $t$ are replaced by node $s^{\text{out}} = s'$ and $t^{\text{in}} = t'$.
- Edges connect $u^{\text{mid}}$ with $u^{\text{in}}$ and $u^{\text{out}}$.
- An edge connects $u^{\text{out}}$ and $v^{\text{in}}$ if $(u, v)$ is an edge of $G$.

**The Subset Sum Problem**

- $SUBSET\_SUM = \{\langle S, t\rangle \mid S = \{x_1, \cdots, x_k\}$ and for some$\{y_1, \cdots, y_l\} \subseteq S,$ we have $\sum y_i = t\}$.

  **Theorem 4.** *$SUBSET\_SUM$ is NP-complete.*

- We show that $3SAT \leq_{\mathrm{P}} SUBSET\_SUM$.

**The Subset Sum Problem (cont.)**

| | 1 | 2 | 3 | 4 | $\cdots$ | $l$ | $c_1$ | $c_2$ | $\cdots$ | $c_k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y_1$ | 1 | 0 | 0 | 0 | $\cdots$ | 0 | 1 | 0 | $\cdots$ | 0 |
| $z_1$ | 1 | 0 | 0 | 0 | $\cdots$ | 0 | 0 | 0 | $\cdots$ | 0 |
| $y_2$ | | 1 | 0 | 0 | $\cdots$ | 0 | 0 | 1 | $\cdots$ | 0 |
| $z_2$ | | 1 | 0 | 0 | $\cdots$ | 0 | 1 | 0 | $\cdots$ | 0 |
| $y_3$ | | | 1 | 0 | $\cdots$ | 0 | 1 | 1 | $\cdots$ | 0 |
| $z_3$ | | | 1 | 0 | $\cdots$ | 0 | 0 | 0 | $\cdots$ | 1 |
| $\vdots$ | | | | | $\ddots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ |
| $y_l$ | | | | | | 1 | 0 | 0 | $\cdots$ | 0 |
| $z_l$ | | | | | | 1 | 0 | 0 | $\cdots$ | 0 |
| $g_1$ | | | | | | | 1 | 0 | $\cdots$ | 0 |
| $h_1$ | | | | | | | 1 | 0 | $\cdots$ | 0 |
| $g_2$ | | | | | | | | 1 | $\cdots$ | 0 |
| $h_2$ | | | | | | | | 1 | $\cdots$ | 0 |
| $\vdots$ | | | | | | | | | $\ddots$ | $\vdots$ |
| $g_k$ | | | | | | | | | | 1 |
| $h_k$ | | | | | | | | | | 1 |
| $t$ | 1 | 1 | 1 | 1 | $\cdots$ | 1 | 3 | 3 | $\cdots$ | 3 |

**FIGURE 7.57**
Reducing *3SAT* to *SUBSET-SUM*

Source: [Sipser 2006]